Fast high-resolution drawing of algebraic curves and surfaces

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Overview

Implicit curve drawing

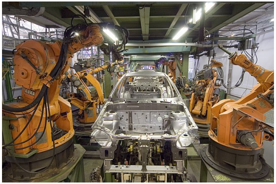
Scientific visualization

Some scientific visualization applications:

- modeling
- medical imaging
- mechanism design

Goal: build an intuition and get an understanding of the data

3D CT reconstruction of distal tibia fracture



Industrial robots from KUKA by Mixabest (CC BY-SA 3.0)

Implicit curve drawing problem

General problem

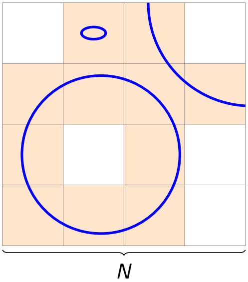
Discrete representation of an implicit curve on a fixed grid

- **Input**:
  - function F
  - resolution N
  - visualization window

  Implicit curve defined as the solution set

$$\{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$$

- **Output**: drawing (set of pixels)



Implicit curve drawing problem

Our focus

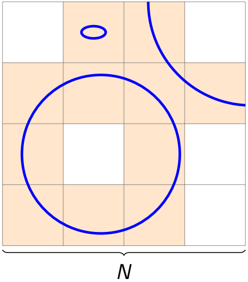
Discrete representation of an **algebraic curve** on a fixed grid

- **Input**:
  - **bivariate polynomial** P of **partial degree** d
  - resolution N
  - **window** $[-1, 1] \times [-1, 1]$

  Algebraic curve defined as the solution set

$$\{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$$

- **Output**: drawing (set of pixels)



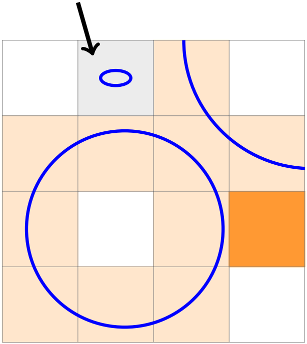
Goal: fast high-resolution drawing of high degree algebraic curves

- $d \approx 100 \quad \longrightarrow d^2 \approx 10,000$ monomials
- $N \approx 1,000$

Correctness of the drawing

For numerical reasons, there may be some:

- **False negative** pixels



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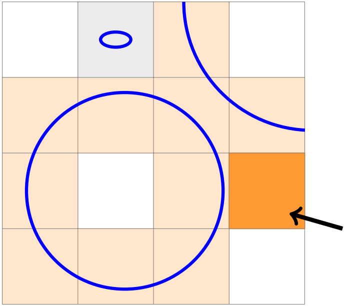
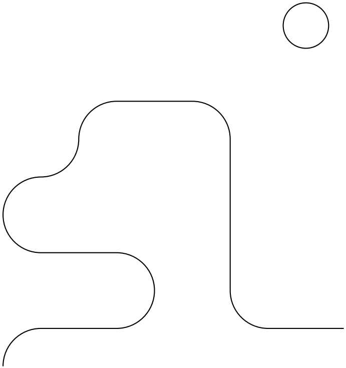


Previous work

Marching squares

The idea

2D variant of the widely used marching cubes algorithm [Lorensen & Cline, 1987]
Implicit curve defined by $P(X, Y) = 0$



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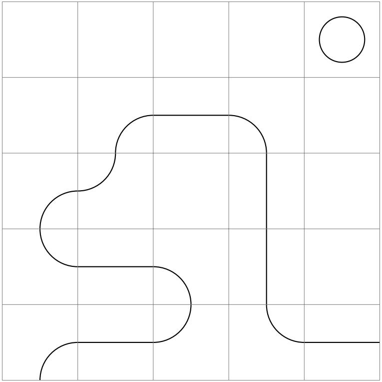

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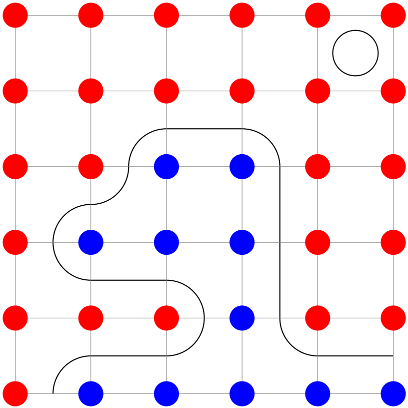

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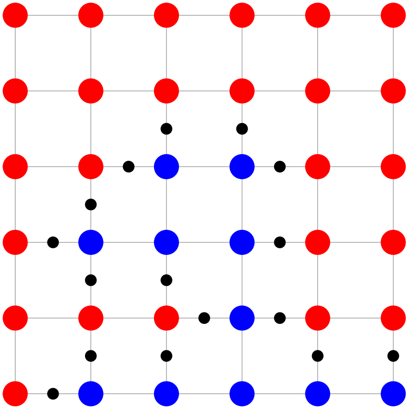

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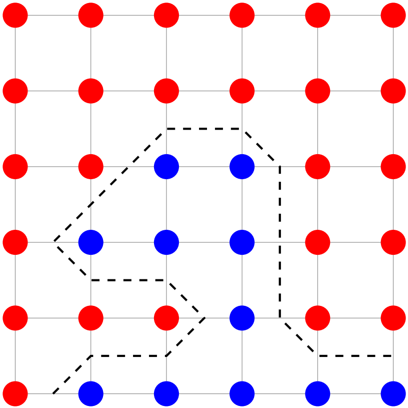

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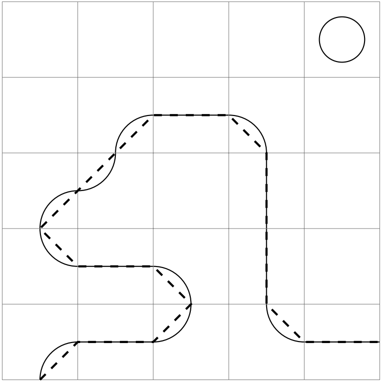

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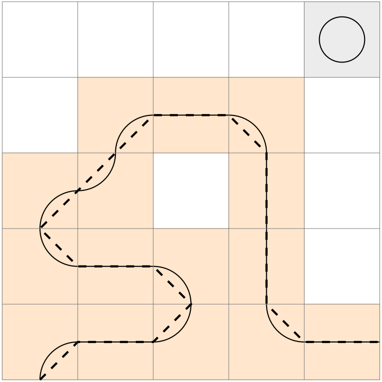


Marching squares

Complexity

Complexity (number of elementary operations)
Naive evaluation

$$\theta(d^2 N^2)$$

d partial degree
N resolution of the grid

Arithmetic complexity of the marching squares

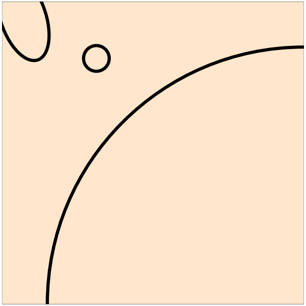
With partial evaluation of $P(x, y)$, assuming $d < N$

$$\theta(dN^2)$$

Slow for high resolutions...
Can we have an algorithm in $O(dN)$?

Adaptive subdivision

Local refinements of the grid



Adaptive subdivision

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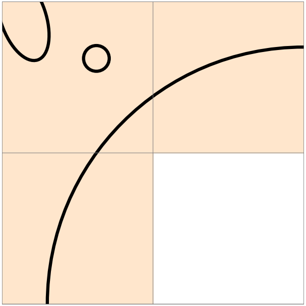

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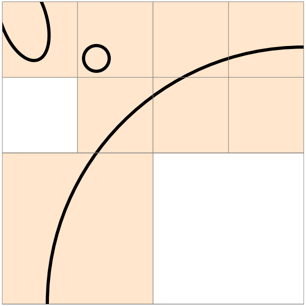

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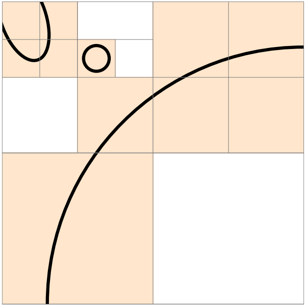

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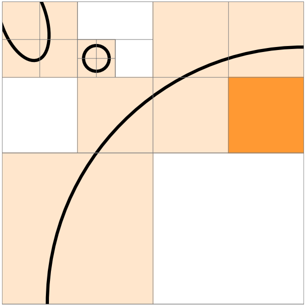
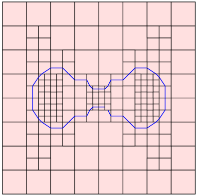


Methods providing topological correctness

Adaptive 2D subdivision with interval arithmetic

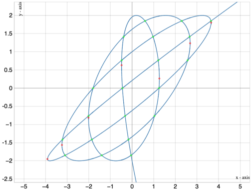
- [Snyder, 1992]
- [Plantinga & Vegter, 2004]
- [Burr et al., 2008]
- [Lin & Yap, 2011]
- ...



[Lin & Yap, 2011]

Cylindrical algebraic decomposition (CAD)

- [Gonzalez-Vega & Necula, 2002]
- [Eigenwillig et al., 2007]
- [Alberti et al., 2008]
- [Cheng et al., 2009]
- [Kobel & Sagraloff, 2015]
- [Diatta et al., 2018]
- ...



https://isotop.gamble.loria.fr/

Our approach

A prerequisite

Interval arithmetic

For $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$,

- $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$
- $I - J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$
- $\ldots$

A prerequisite

Interval arithmetic

For $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$,

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- $I - J = [\underline{I} - \overline{J}, \overline{I} - \underline{J}]$
- ...

Evaluation of the function $f(X) = X^2 - X = (X - 1)\,X$ on the interval $[0, 2]$

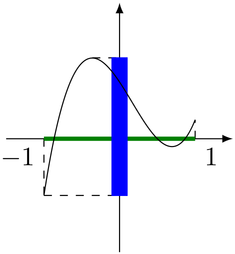
- $[0, 2]^2 - [0, 2] = [0, 4] - [0, 2] = [-2, 4]$
- $([0, 2] - 1) \cdot [0, 2] = [-1, 1] \cdot [0, 2] = [-2, 2]$

Interval arithmetic

Inclusion property

$$P(X) = 2X^3 - X^2 - 1.5X + 0.75$$

How to compute $P(I)$ for $I = [-1, 1]$?



x	-1		$x_1 = \frac{1-\sqrt{10}}{6}$		$x_2 = \frac{1+\sqrt{10}}{6}$		1
$P'(x)$		$+$	0	$-$	0	$+$	
$P(x)$	$P(-1)$	$\nearrow$	$P(x_1)$	$\searrow$	0 $P(x_2)$	$\nearrow$	$P(1)$

$P(I) = [-0.75, 1.06\ldots]$

Interval arithmetic

Inclusion property

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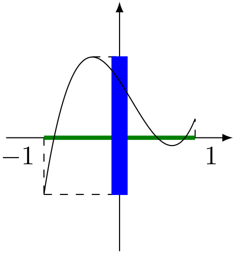
How to compute $P(I)$ for $I = [-1, 1]$?



$$\square P(I) = 2[-1,1]^3 - [-1,1]^2 - 1.5[-1,1] + 0.75$$
$$= [-5.25, 5.25]$$

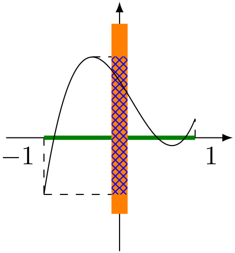
$$P(I) = [-0.75, 1.06\ldots]$$

Interval arithmetic

Inclusion property

$$P(X) = 2X^3 - X^2 - 1.5X + 0.75$$

How to compute $P(I)$ for $I = [-1, 1]$?



$$P(I) = [-0.75, 1.06\ldots]$$

$$\begin{aligned}\Box P(I) &= 2[-1,1]^3 - [-1,1]^2 - 1.5[-1,1] + 0.75\\ &= [-5.25, 5.25]\end{aligned}$$

With Horner's scheme:

$$\begin{aligned}\Box P(I) &= ((2[-1,1] - 1)[-1,1] - 1.5)[-1,1] + 0.75\\ &= [-3.75, 5.25]\end{aligned}$$

$$P(I) \subseteq \Box P(I)$$

Interval arithmetic

Convergence property

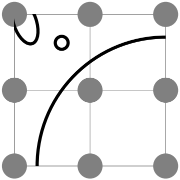
Convergence at a point

With $x \in [a, b]$

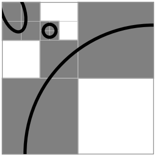
$$\lim_{[a,b] \longrightarrow [x,x] = \{x\}} \square P([a, b]) = P(x)$$

Our approach: guaranteed intersection with the grid

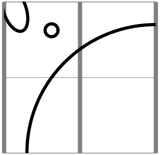
Marching squares



Adaptive subdivision



New approach: evaluation along fibers



⇒ Make it fast and provide some guarantees

Two algorithms

Edge drawing

- *evaluation in X*
  Chebyshev nodes
  multipoint evaluation with IDCT
- *subdivision in Y*
  naive root finding method

Guarantees
False positive and false negative pixels

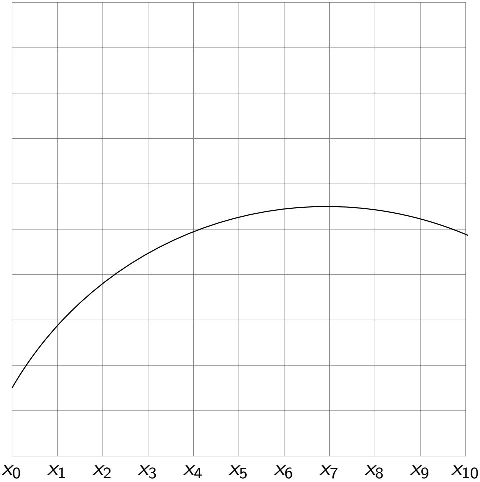
Pixel drawing

- *evaluation in X*
  Chebyshev nodes
  multipoint evaluation with IDCT
  Taylor approximation
- *subdivision in Y*
  naive root finding method

Guarantees
False positive pixels *only*

Subdivisions along a fiber

$P(x_k, Y) = \sum a_j Y^j$



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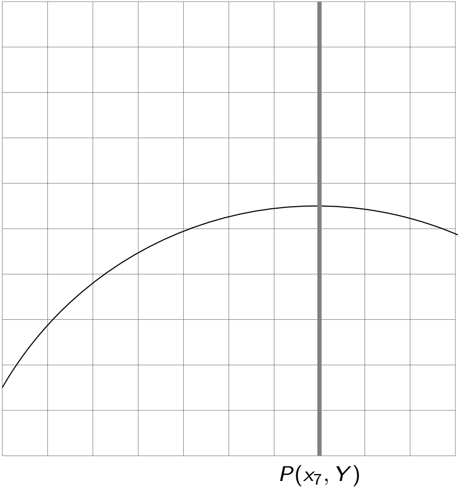

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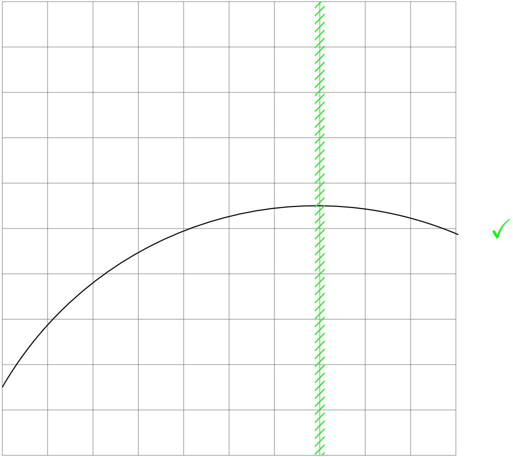

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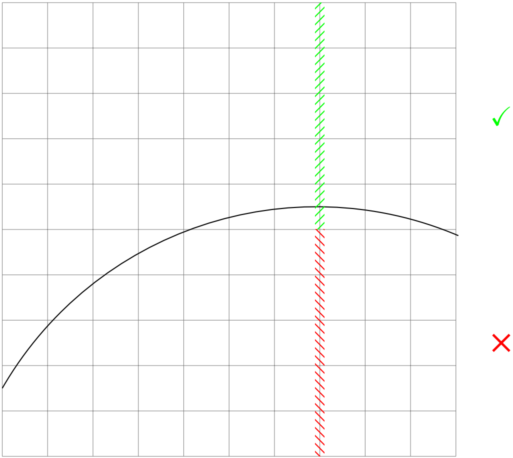

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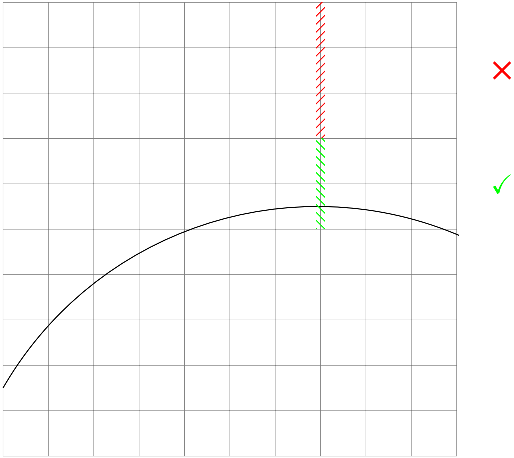
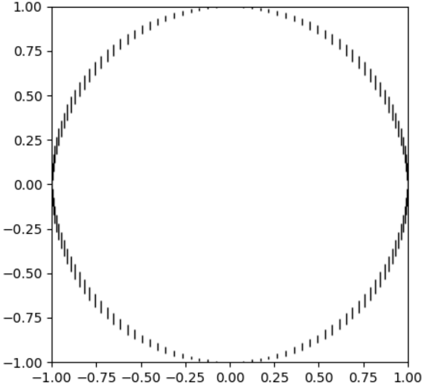


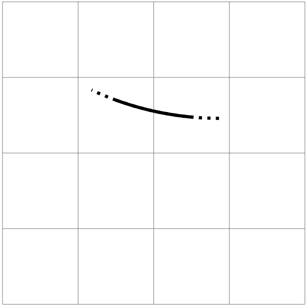
An example

$$X^2 + Y^2 - 1 = 0$$



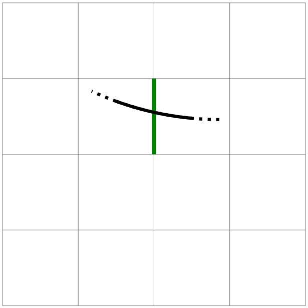
Resolution $N = 64$

Pixel lighting
Edge drawing



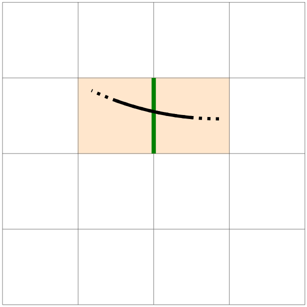
Pixel lighting
Edge drawing

- Detect a crossing between two consecutive nodes of the grid



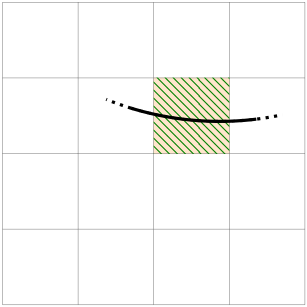
Pixel lighting
Edge drawing

- Detect a crossing between two consecutive nodes of the grid
- Light the adjacent pixels



Pixel lighting
Pixel drawing

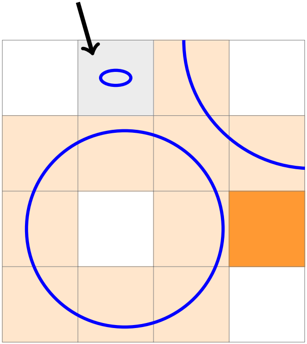
- Detect a crossing in pixel of the grid
- Light that pixel



False positive and false negative pixels
Edge drawing

Some incorrect pixels:

- **False negative** when a connected component lies inside of a pixel



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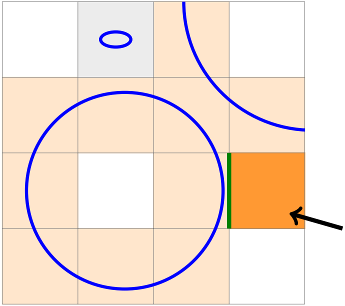
Some incorrect pixels:

- **False negative** when a connected component lies inside of a pixel
- **False positive** when the evaluation on an edge of a pixel is close to zero
  That occurs for a segment S when

$$0 \in \Box P(S) + [-E, E]$$



Certification of segments that are not crossed:

$$0 \notin \Box P(S) + [-E, E]$$
$$\Downarrow$$
$$0 \notin P(S)$$

False positive and false negative pixels
Pixel drawing

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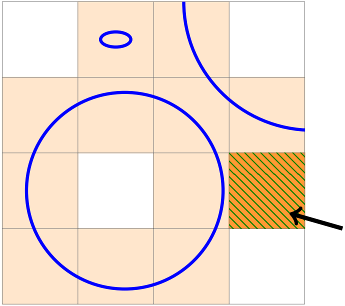

Certification of segments that are not crossed:

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Fast multipoint evaluation

A prerequisite to fast multipoint evaluation

Chebyshev polynomials

Definition

The Chebyshev polynomials (T_k) verify $\forall k \in \mathbb{N}$, $T_k(\cos\theta) = \cos(k\theta)$

The first three Chebyshev polynomials

$$\begin{aligned} \cos(0\cdot\theta) &= 1 & T_0 &= 1 \\ \cos(1\cdot\theta) &= \cos(\theta) & T_1 &= X \\ \cos(2\cdot\theta) &= 2\cos(\theta)^2 - 1 & T_2 &= 2X^2 - 1 \end{aligned}$$

A prerequisite to fast multipoint evaluation

Chebyshev polynomials

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Lemma

An arbitrary polynomial p of degree d can be written in terms of the Chebyshev polynomials:

$$p(X) = \sum_{k=0}^{d} \alpha_k T_k(X)$$

A prerequisite to fast multipoint evaluation

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Lemma

For $N \in \mathbb{N}$, a polynomial p of degree d can be evaluated on the Chebyshev nodes $(c_n)_{0 \le n \le N-1}$ using the IDCT*:*

$$(p(c_n))_{0 \le n \le N-1} = \frac{1}{2}(\alpha_0, \ldots, \alpha_0) + \mathrm{IDCT}((\alpha_k)_{0 \le k \le N-1})$$

A prerequisite to fast multipoint evaluation

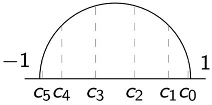
Chebyshev nodes

Definition

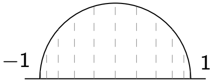
For $N \in \mathbb{N}$, the Chebyshev nodes are

$$c_n = \cos\left(\frac{2n+1}{2N}\pi\right), \quad n = 0, \ldots, N-1$$

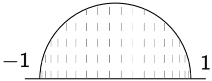
They are the roots of T_N



$N = 6$



$N = 11$



$N = 20$

Inverse Discrete Cosine Transform

Inverse Discrete Cosine Transform (IDCT): $\alpha_k \to x_n$

$$x_n = \frac{1}{2}\alpha_0 + \sum_{k=1}^{N-1} \alpha_k \ \cos\left[\frac{\pi k(2n+1)}{2N}\right]$$

$$\begin{tikzcd}
(\alpha_k) \arrow[r, dashed, "\textit{linear transformation}"] \arrow[rrr, dotted, bend left, "\text{IDCT}"] & (V_k) \arrow[r, "\text{FFT}"] & (v_k) \arrow[r, dashed, "\textit{linear transformation}"] & (x_k)
\end{tikzcd}$$

$\Rightarrow$ Fast thanks to the Fast Fourier Transform (FFT) algorithm in $O(N \log_2 N)$

[Makhoul, 1980]

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\end{tikzcd}$$

$\Rightarrow$ Fast thanks to the Fast Fourier Transform (FFT) algorithm in $O(N \log_2 N)$

[Makhoul, 1980]

$$p(c_n) = \sum_{k=0}^{N-1} \alpha_k T_k\left(\cos\left(\frac{2n+1}{2N}\pi\right)\right)$$

Inverse Discrete Cosine Transform

Inverse Discrete Cosine Transform (IDCT): $\alpha_k \to x_n$

$$x_n = \frac{1}{2}\alpha_0 + \sum_{k=1}^{N-1} \alpha_k \ \cos\left[\frac{\pi k(2n+1)}{2N}\right]$$

$$\begin{tikzcd}
(\alpha_k) \arrow[r, dashed, "\textit{linear transformation}"] \arrow[rrr, dotted, bend left, "\mathrm{IDCT}"] & (V_k) \arrow[r, "\mathrm{FFT}"] & (v_k) \arrow[r, dashed, "\textit{linear transformation}"] & (x_k)
\end{tikzcd}$$

$\Rightarrow$ Fast thanks to the Fast Fourier Transform (FFT) algorithm in $O(N \log_2 N)$

[Makhoul, 1980]

$$p(c_n) = \sum_{k=0}^{N-1} \alpha_k T_k\left(\cos\left(\frac{2n+1}{2N}\pi\right)\right) = \sum_{k=0}^{N-1} \alpha_k \cos\left[\frac{\pi k(2n+1)}{2N}\right]$$

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$$(p(c_n))_{0 \le n \le N-1} = \frac{1}{2}(\alpha_0, \ldots, \alpha_0) + \mathrm{IDCT}((\alpha_k)_{0 \le k \le N-1})$$

Error of the IDCT

[Makhoul, 1980] and [Brisebarre et al., 2020, Theorem 3.4] yield

Theorem (H., Moroz, Pouget, 2022)

Assume radix-2, precision-p arithmetic, with rounding unit $u = 2^{-p}$ *. Let* $\widehat{x}$ *be the computed* 2^n*-point IDCT of* $\alpha \in \mathbb{C}^{2^n}$*, and let* x *be the exact value. Then*

$$\|\widehat{x} - x\|_\infty = n\|\alpha\|_\infty O(u).$$

Table: IDCT error bounds for $p = 53$ (double precision)

$N = 2^n$	$1,024$	$2,048$	$4,096$	$8,192$	$16,384$	$32,768$
$\|\widehat{x} - x\|_\infty / \|\alpha\|_\infty$	7.97e-15	8.84e-15	9.72e-15	1.06e-14	1.15e-14	1.23e-14

Algorithms

General idea: edge enclosure

Illustration

$$P(X,Y) = \sum \left(\sum a_{i,j} X^i \right) Y^j = \sum p_j(X) Y^j$$

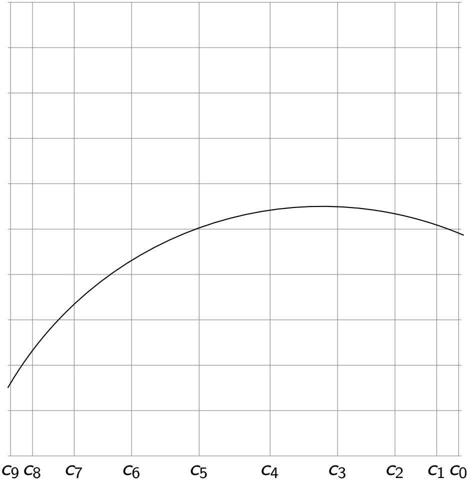
$$p_j(X) = \sum a_{i,j} X^i = \sum \alpha_{i,j} T_i(X)$$

$$(p_j(c_n))_{0 \leq n \leq N-1} = \frac{1}{2}(\alpha_{0,j}, \ldots, \alpha_{0,j}) + \mathrm{IDCT}((\alpha_{k,j})_{0 \leq k \leq N-1})$$

General idea: edge enclosure

Illustration

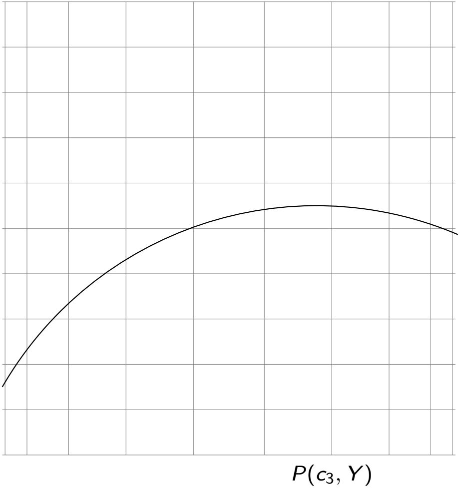
$P(c_n, Y) = \sum p_j(c_n) Y^j$



General idea: edge enclosure

Illustration

$P(c_3, Y) = \sum p_j(c_3) Y^j$



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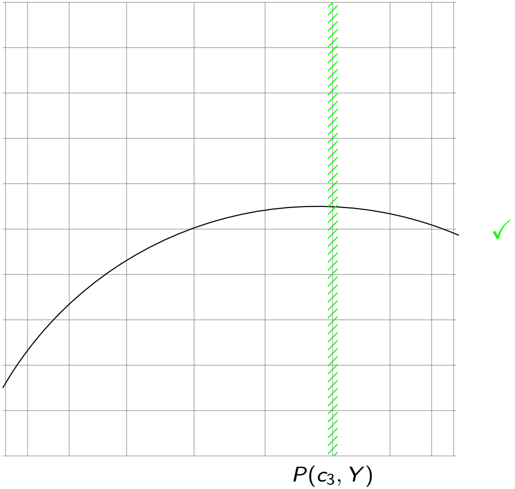

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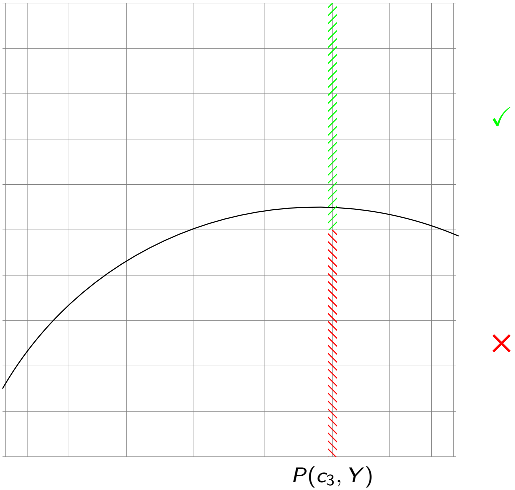

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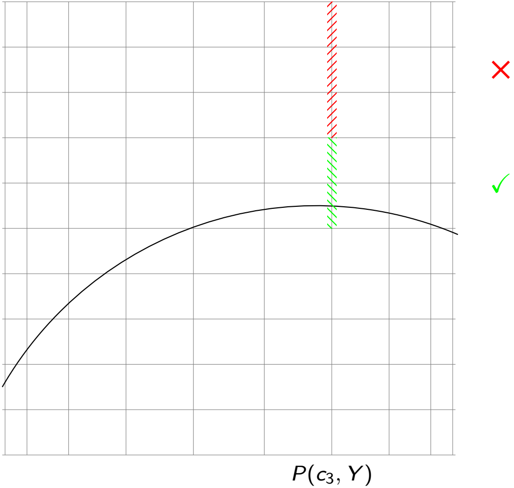
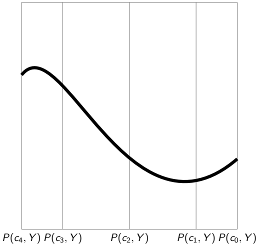


An edge enclosing algorithm

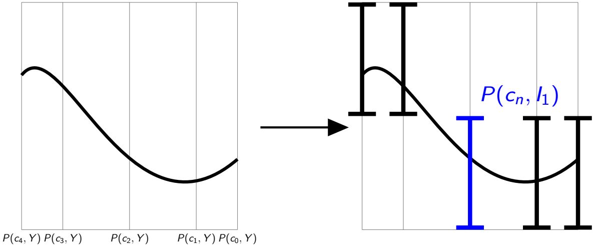


IDCT multipoint evaluation in X at $c_0, c_1 \ldots$

subdivision in Y

IDCT multipoint evaluation of the partial polynomials of $P(X,Y) = \sum p_j(X) Y^j$

An edge enclosing algorithm



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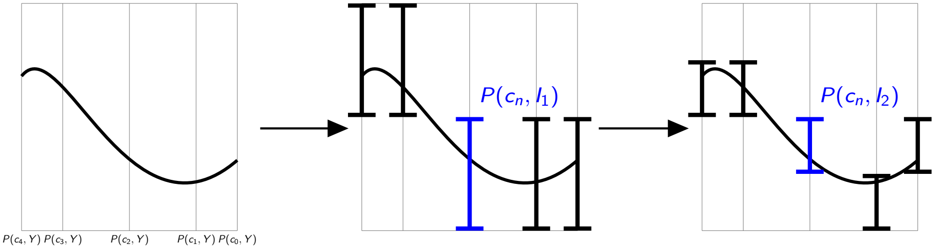


IDCT multipoint evaluation in X at $c_0, c_1 \dots$

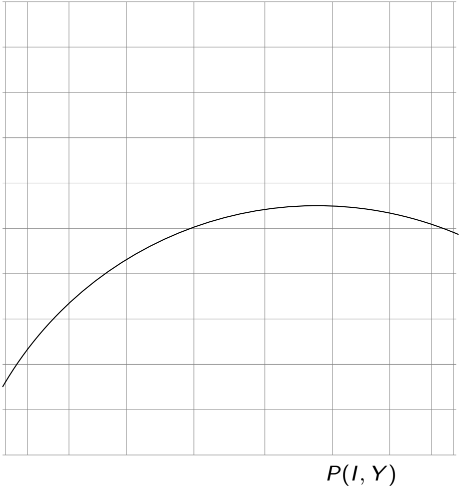
subdivision in Y

IDCT multipoint evaluation of the partial polynomials of $P(X,Y) = \sum p_j(X)Y^j$

General idea: pixel enclosure

Illustration

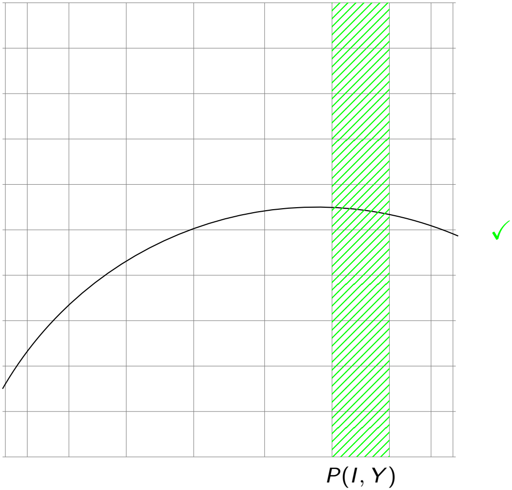
$P(I, Y) = \sum p_j(I) Y^j$



General idea: pixel enclosure

Illustration

$P(I, Y) = \sum p_j(I) Y^j$



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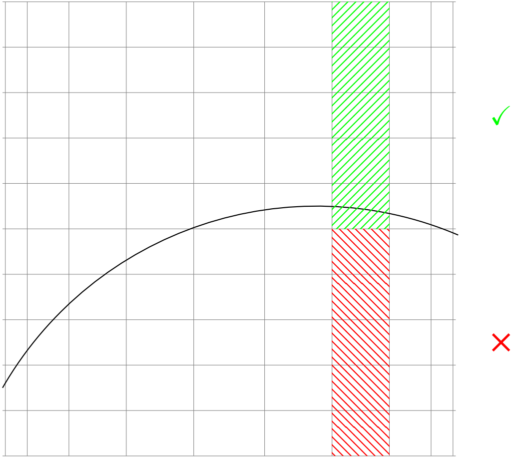

$P(I, Y)$

General idea: pixel enclosure

Illustration

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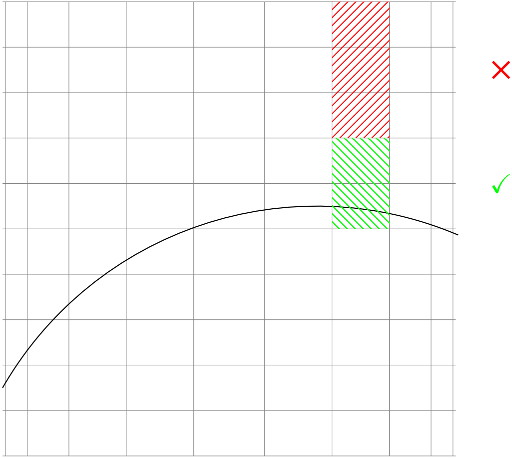
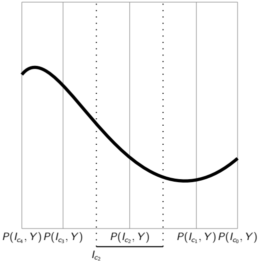


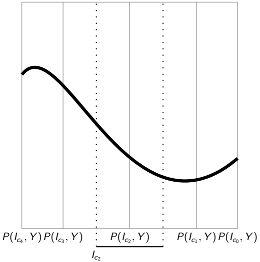
A pixel enclosing algorithm



IDCT multipoint evaluation in X
around $c_0, c_1 \ldots$

subdivision in Y

A pixel enclosing algorithm



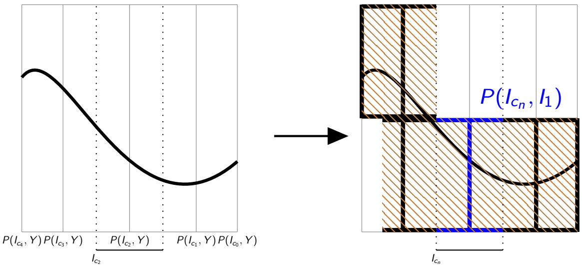
IDCT multipoint evaluation +
Taylor approximation in X

subdivision in Y

Taylor expansion of the partial polynomials of $P(X, Y) = \sum p_j(X) Y^j$

$$\left| p(c_n + r) - \left(p(c_n) + r p'(c_n) + \cdots + \frac{r^m}{m!} p^{(m)}(c_n) \right) \right| \leq \max_{I_{c_n}} \left| p^{(m+1)} \right| \frac{|r|^{(m+1)}}{(m+1)!}$$

A pixel enclosing algorithm



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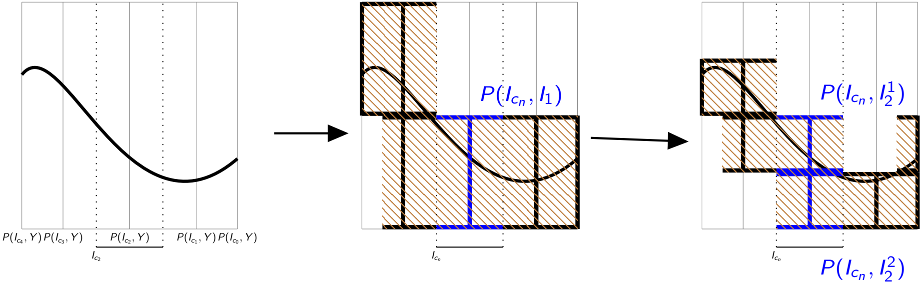

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Complexities

Arithmetic complexities

multipoint evaluation and subdivision	$O(\ \ d^3 + \ \ dN\log_2(N) + dNT)$
multipoint Taylor approximation and subdivision	$O(md^3 + mdN\log_2(N) + dNT)$

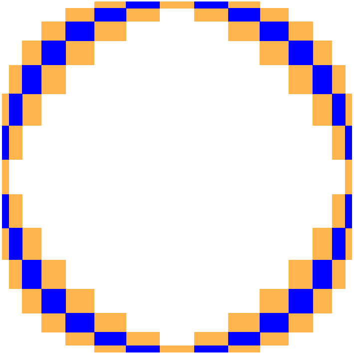
d partial degree
N resolution
T maximum number of nodes of the subdivision trees over all vertical fibers / stripes

With a constant number of branches in the window, we expect $T = O(\log_2(N))$

Experiments

Pixel classification

- crossed: blue
- not crossed: white
- undecided: yellow

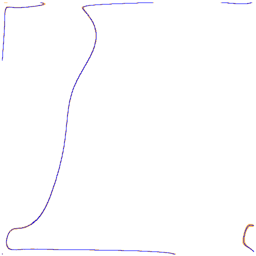


Drawing for two families of polynomials

Experiments on smooth curves $\longrightarrow$ random polynomials
$\xi_{i,j}$: random coefficients in $[-100, 100]$

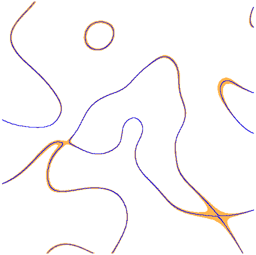
Kac polynomial

$$P(X,Y) = \sum_{i+j=0}^{d} \xi_{i,j} X^i Y^j$$



Kostlan-Shub-Smale (KSS) polynomial

$$P(X,Y) = \sum_{i+j=0}^{d} \sqrt{\frac{d!}{i!j!(d-i-j)!}} \xi_{i,j} X^i Y^j$$



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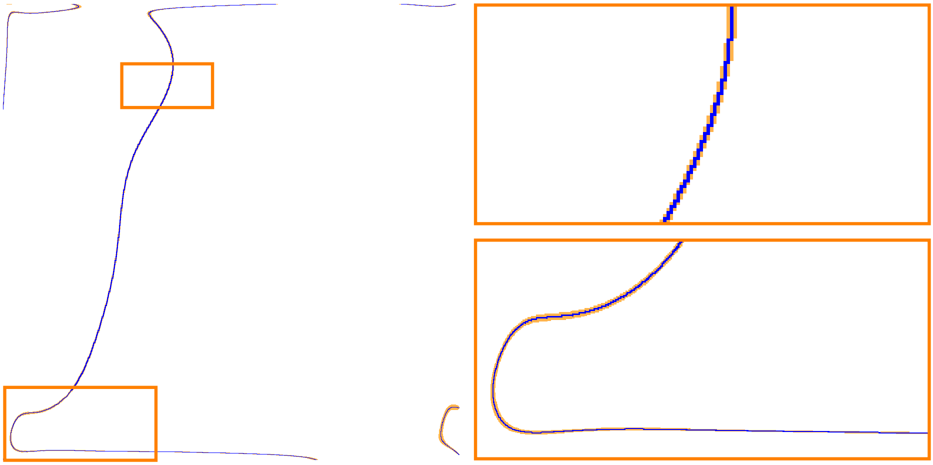


(a)

(b)

Figure: Kac polynomial of degree $d = 110$ at a resolution $N = 1,024$, $\frac{b}{b+y} = 24\%$

Drawing for two families of polynomials

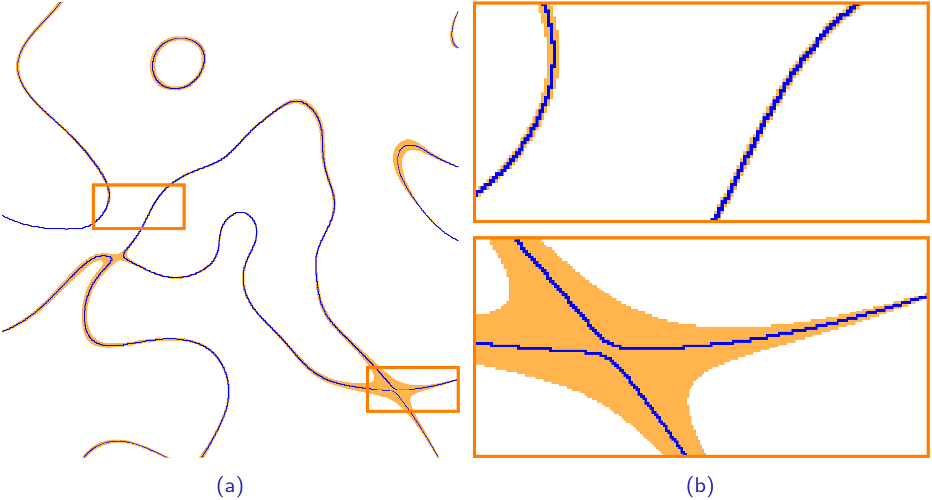


Figure: KSS polynomial of degree $d = 40$ at a resolution $N = 1,024$, $\frac{b}{b+y} = 19\%$

Comparison to state-of-the-art software

Our methods

- edge drawing $\to$ curve enclosing edges — false positive and false negative
- pixel drawing $\to$ curve enclosing pixels — false positive

Some similar methods

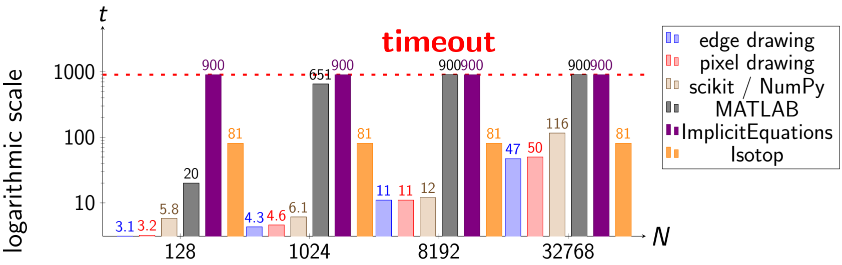
- scikit / NumPy $\to$ marching squares — false negative
- MATLAB $\to$ could not find the method used — false negative?
- ImplicitEquations $\to$ 2D adaptive subdivision — false positive

A topologically correct method

- Isotop $\to$ cylindrical algebraic decomposition

Timing

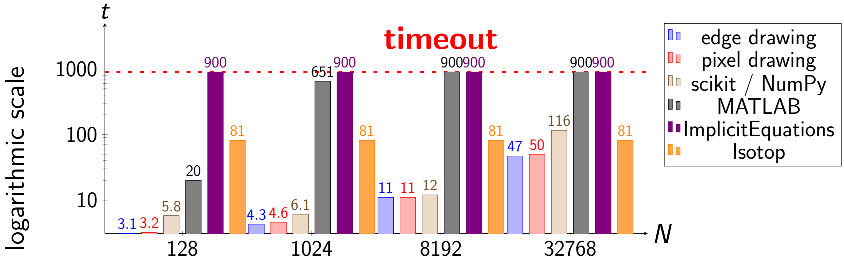
Comparison for a polynomial



Computation times for a **Kac** polynomial of degree 40 (in seconds)

Timing

Comparison for a polynomial



Computation times for a **Kac** polynomial of degree 40 (in seconds)

scikit: $O(dN^2)$

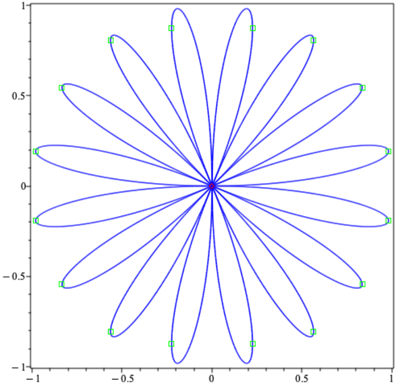
Our methods: $O(dNT)$
as expected $T = O(\log_2(N))$

no guarantee
slow when d and N are large

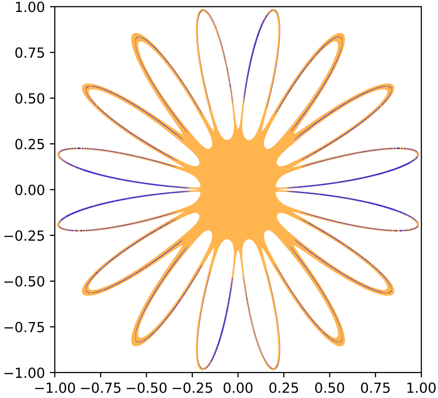
guarantees
fast when d and N are large

Output for a singular curve

Curve: $\mathrm{dfold}_{8,1}$ from Challenge 14 of Oliver Labs[13][37] ($d = 18$)



Isotop



Pixel drawing

Conclusion

Contributions

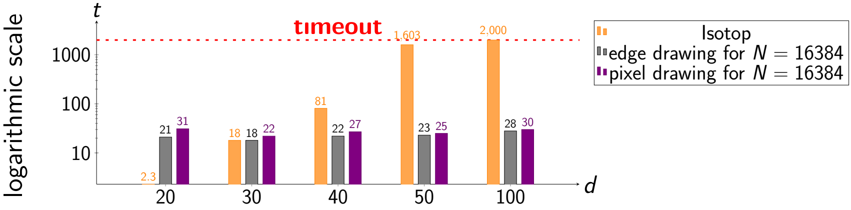
- Two algorithms
  - enclosure of the edges
  - enclosure of the pixels
- Fast implicit curve and surface algorithms for high resolutions: faster than marching squares and marching cubes
- Better guarantees on the drawing than marching squares
- Ability to handle high degrees ($d > 20$) and high resolutions ($N > 1000$)

Future work

- Can the thickness of the drawing be controlled?
- Could we have a faster subdivision with other root finding methods?
- Can the multipoint evaluation improve Plantinga and Vegter's algorithm?

Timing

A CAD approach: Isotop



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