Reinforced Set Projection Algorithm

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ENSTA Bretagne, Lab-STICC

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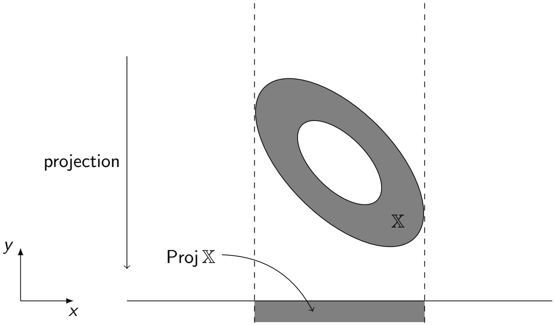
Set description

Set projection

$$\begin{aligned} \mathrm{Proj} : \mathbb{R}^m &\to \mathbb{R}^n \\ \mathbb{X} &\mapsto \mathrm{Proj}\,\mathbb{X} \end{aligned} \qquad n < m$$

$$\mathrm{Proj}\,\mathbb{X} = \{(x_1, \ldots, x_n) \mid \exists (x_{n+1}, \ldots, x_m), (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \in \mathbb{X})\}$$

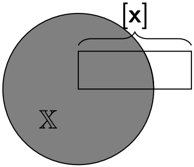


Reinforced Set Projection Algorithm

Goal:

- describe the projection with interval arithmetic
- propose an algorithm (*better than the current one*)

Set description with interval arithmetic: contractors



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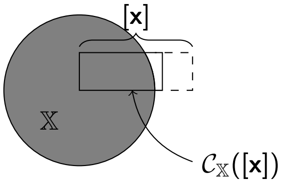

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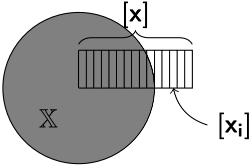


A naive contractor implementation

- bisection
- evaluation of each small box

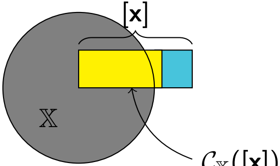
$$f(x,y) = x^2 + y^2 - 1$$

$$\mathbb{X} = \{(x,y) \,|\, f(x,y) \leq 0\}$$

$f([\mathbf{x_i}]) > 0 \to$ outside of $\mathbb{X}$

- merge the boxes that are not clearly outside

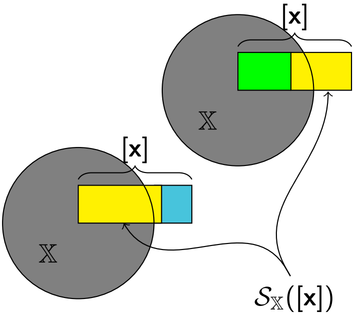
Set description with interval arithmetic: contractors



$\mathcal{C}_{\mathbb{X}}$: contractor for the set $\mathbb{X}$

$\mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \subset [\mathbf{x}]$	contractance
$\mathcal{C}_{\mathbb{X}}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X}$	correctness

Set description with interval arithmetic: separators



$\mathcal{S}_{\mathbb{X}}$: separator for the set $\mathbb{X}$
$\mathcal{S}_{\mathbb{X}}([\mathbf{x}]) = ([\mathbf{x}_1], [\mathbf{x}_2])$ and $[\mathbf{x}_1] \cup [\mathbf{x}_2] = [\mathbf{x}]$

$$[\mathbf{x}_1] \subset [\mathbf{x}]$$
$$[\mathbf{x}_1] \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X}$$

$$[\mathbf{x}_2] \subset [\mathbf{x}]$$
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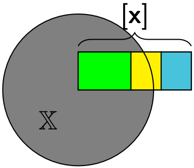

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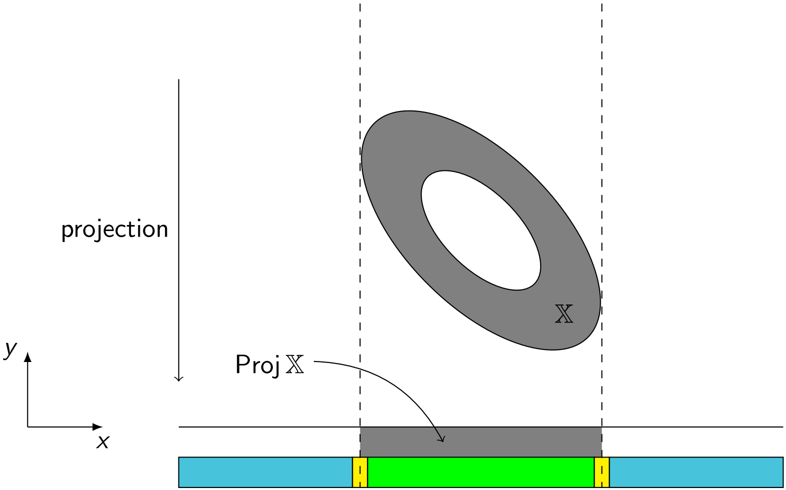
Set projection

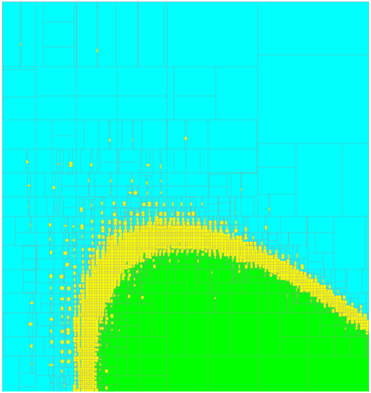
Set projection separator



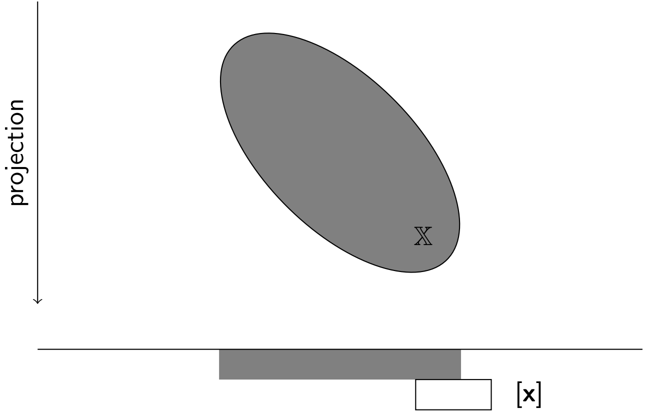
SepProj in the Codac library

$$\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3 \,|\, 2x^2 + 2.2xy + xz + y^2 + z^2 \leq 10\}$$

Projection onto the *xy*-plane: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$



Current projection algorithm: its aim



What we have
$\mathcal{S}_{\mathbb{X}} = \mathcal{C}_{\mathbb{X}}, \mathcal{C}_{\overline{\mathbb{X}}}$

What we want
$\mathcal{S}_{\mathsf{Proj}\,\mathbb{X}}$

We construct f such that
$\mathcal{S}_{\mathsf{Proj}\,\mathbb{X}} = f(\mathcal{S}_{\mathbb{X}})$

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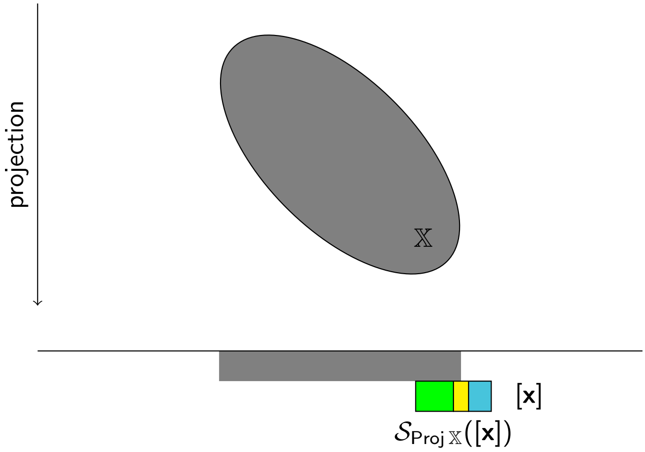

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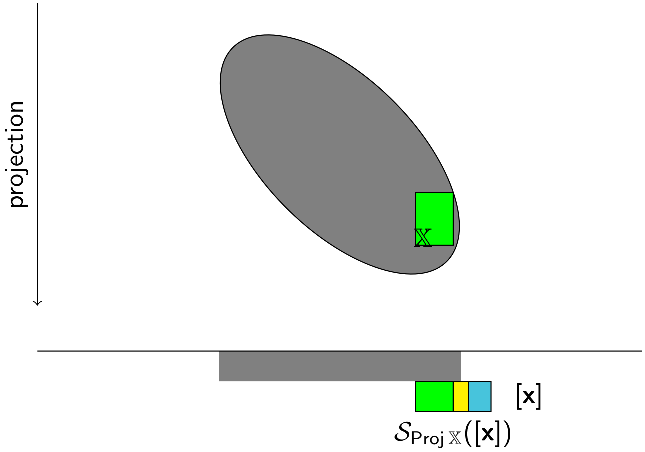

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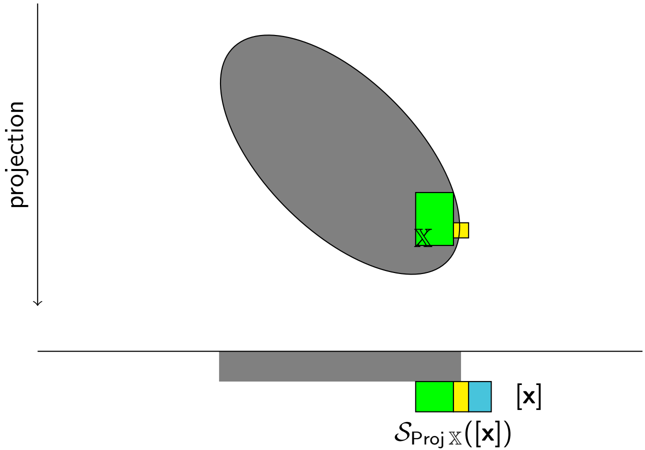

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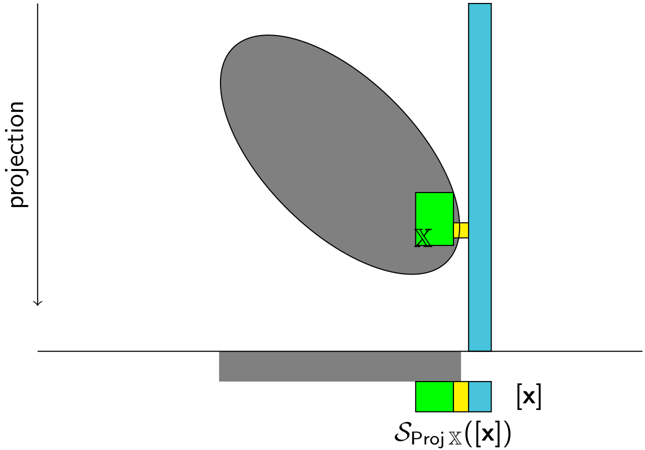

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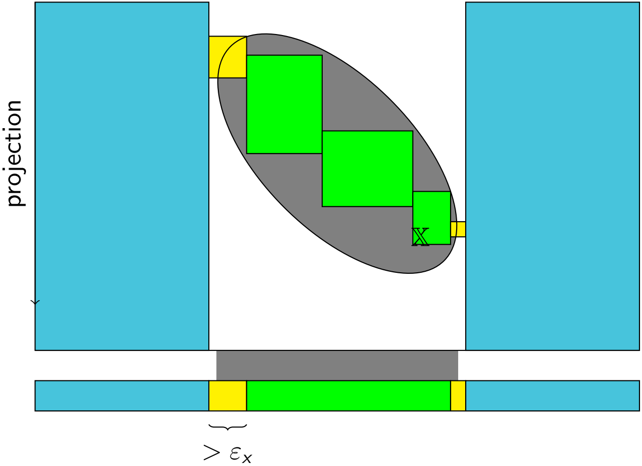
Paving the projection space



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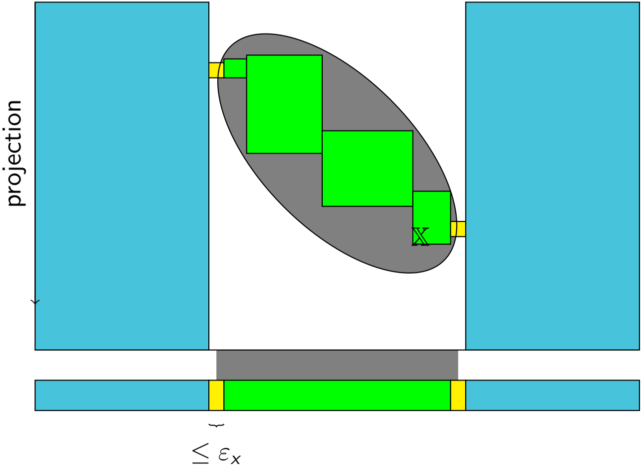

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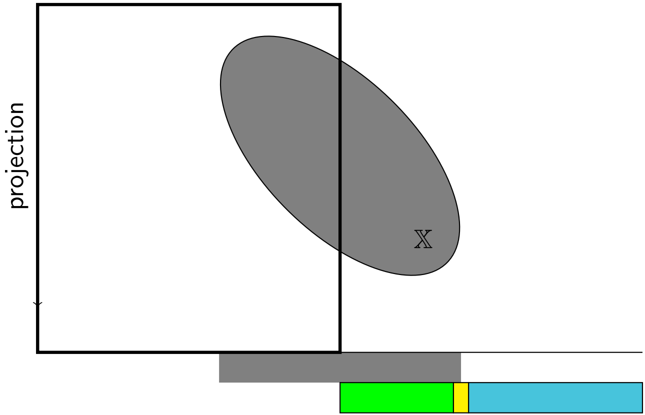
Current projection algorithm: its main steps



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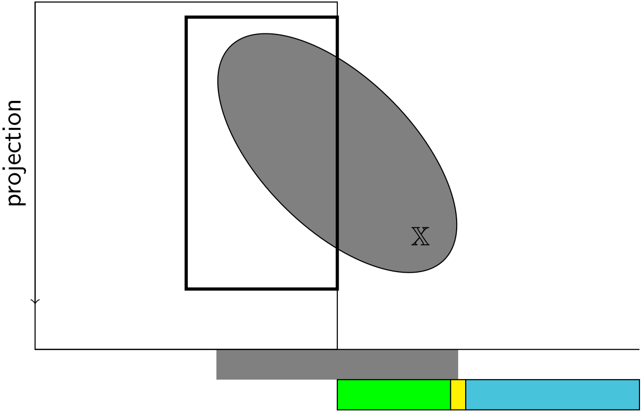

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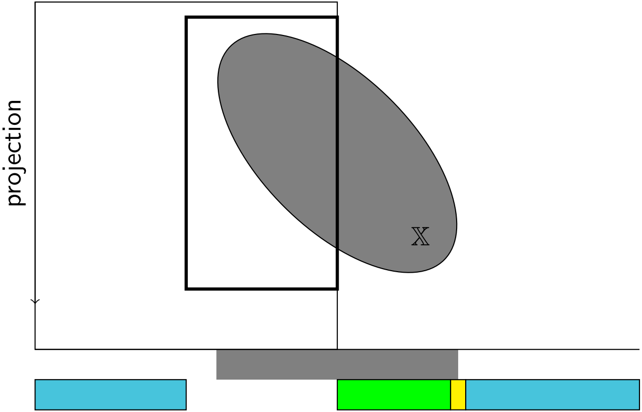

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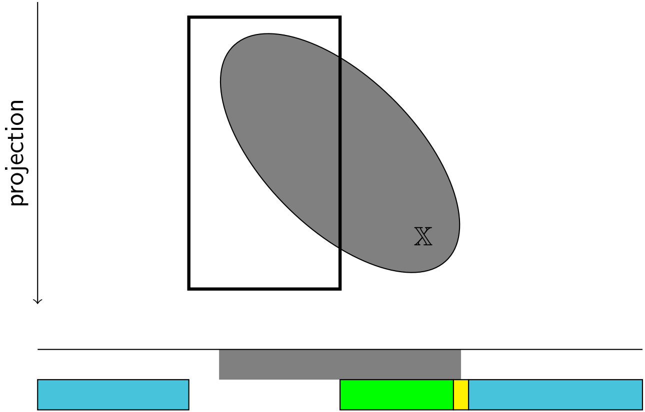

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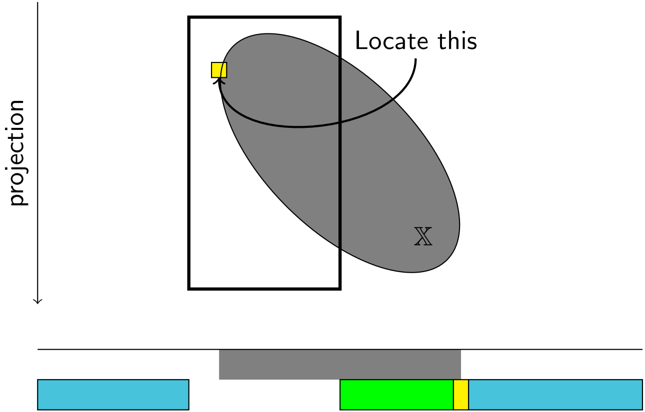

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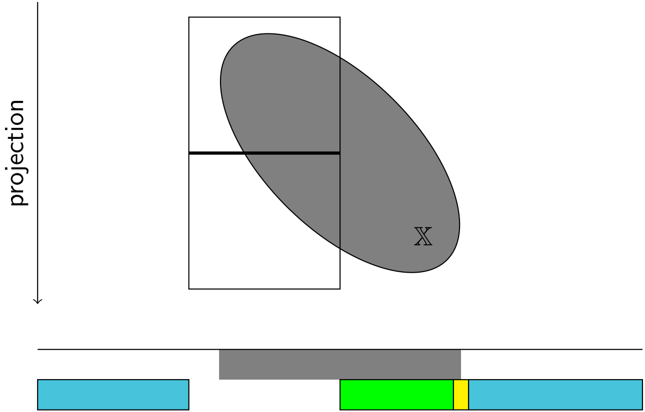

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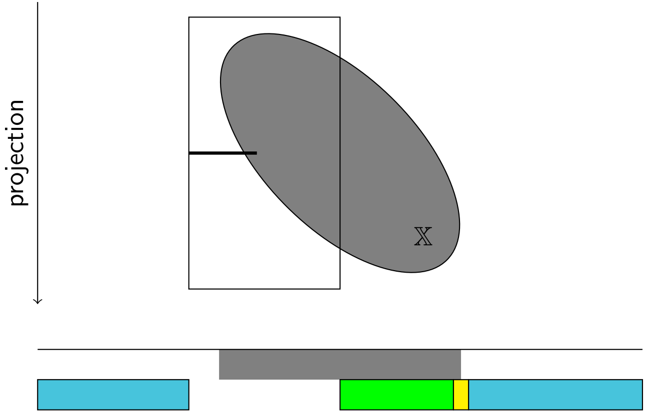

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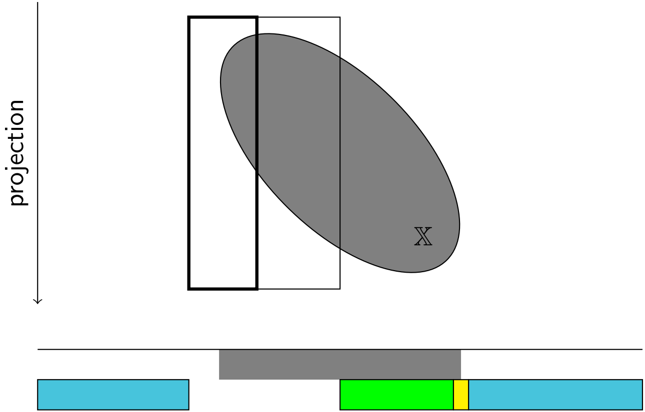

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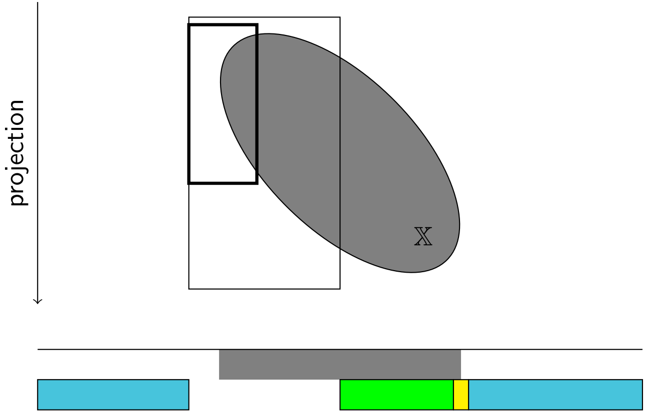

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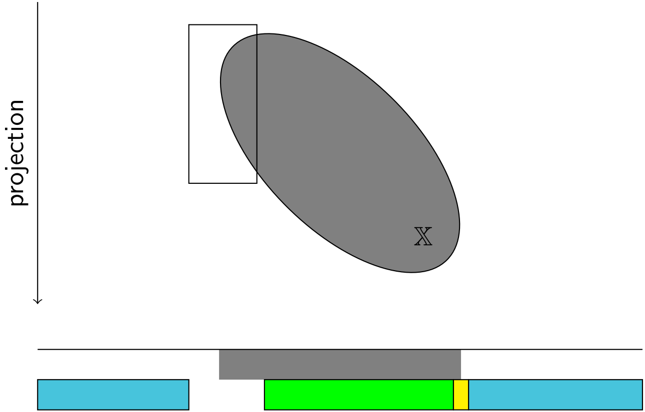

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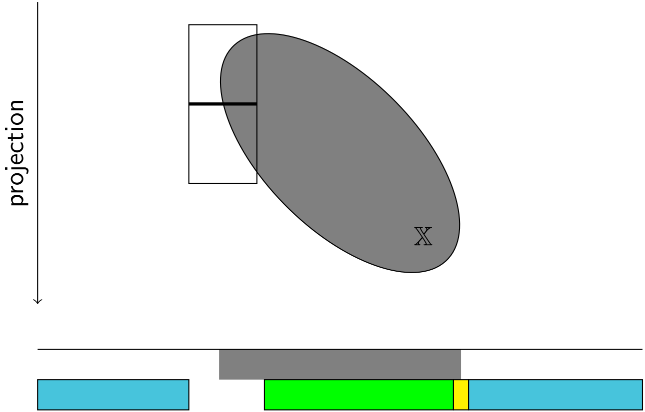

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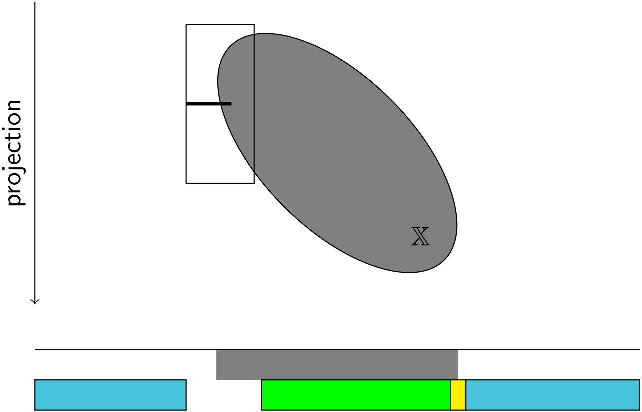

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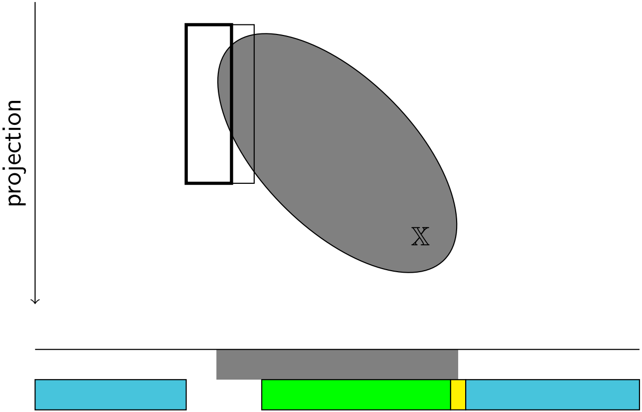

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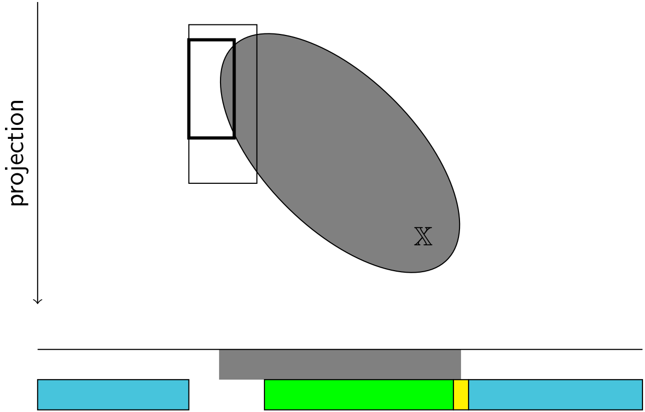

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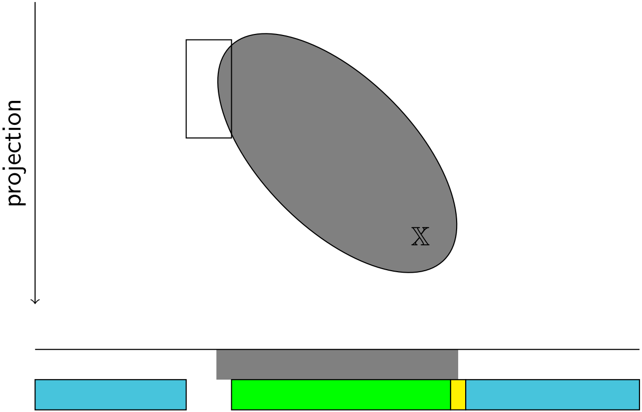

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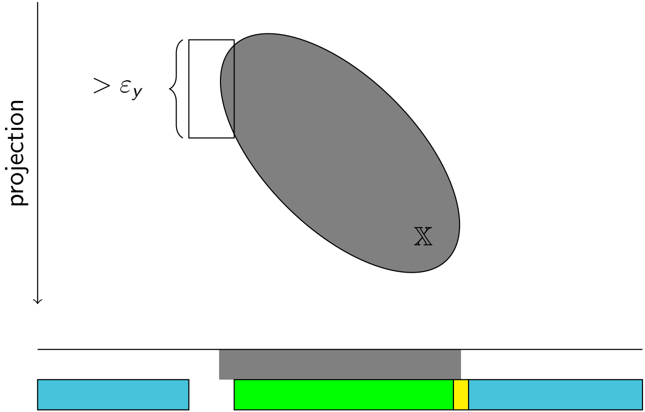

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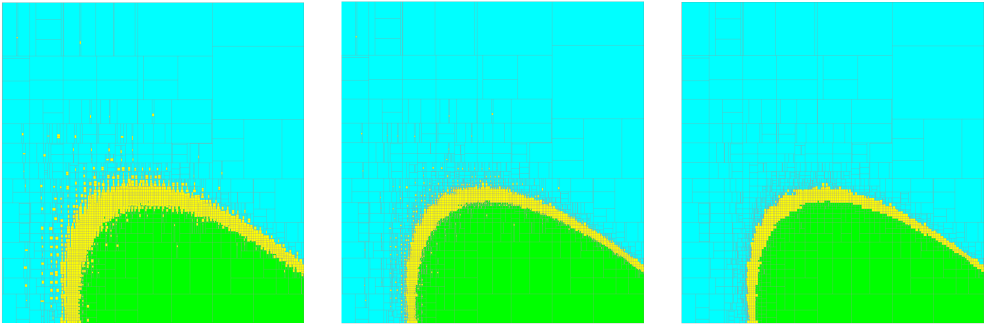
Back to the example

$$\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3 \,|\, 2x^2 + 2.2xy + xz + y^2 + z^2 \leq 10\}$$

Projection onto the xy-plane: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\varepsilon_{xy} = 0.03$$

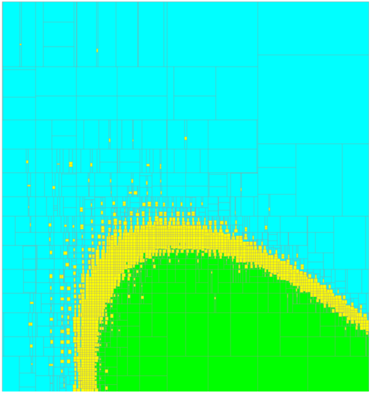


$\varepsilon_z = 0.03$ $\varepsilon_z = 0.015$ $\varepsilon_z = 0.003$

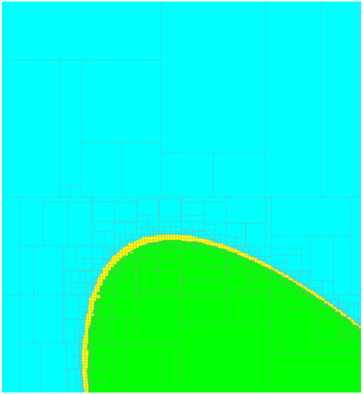
Comparison to the new approach

$$\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3 \,|\, 2x^2 + 2.2xy + xz + y^2 + z^2 \leq 10\}$$

Projection onto the xy-plane: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

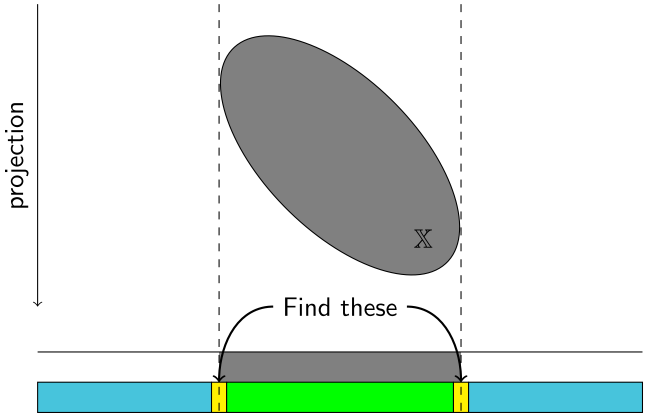


`SepProj`



New approach

Reinforcing the set projection



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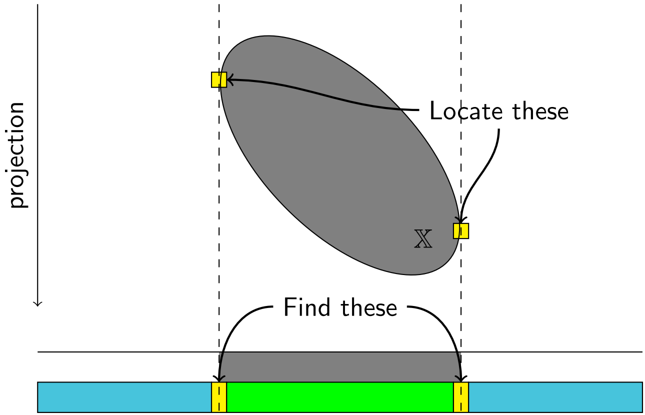
Reinforcing the set projection



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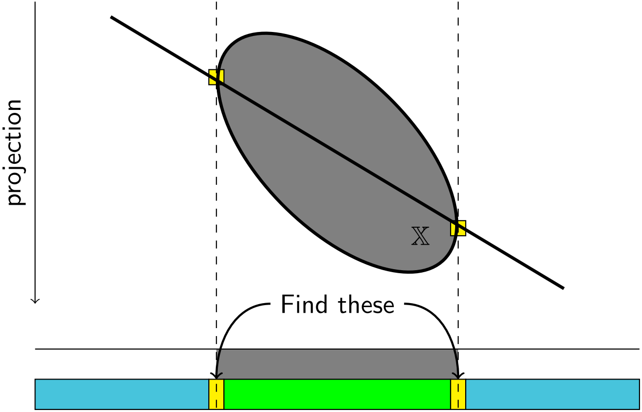
Reinforcing the set projection

Assuming that $\mathbb{X}$ is a **differentiable** set



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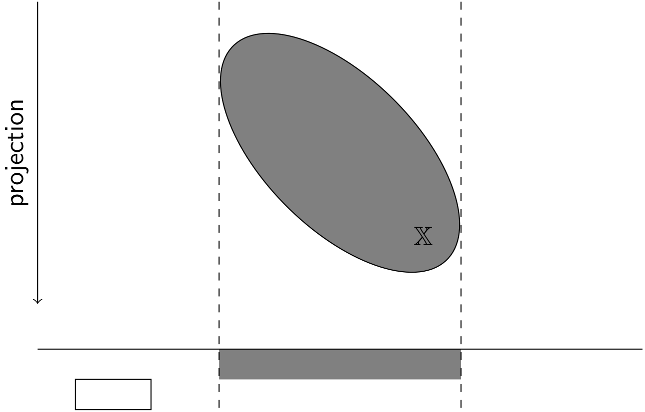
Reinforced set projection: a new paving algorithm



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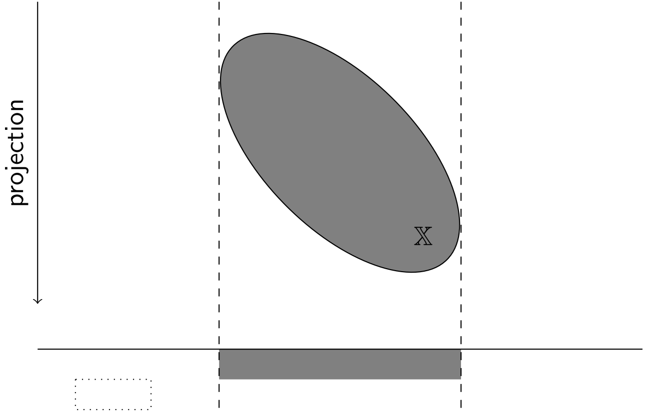
1. Contraction
2. Color from neighbors
   or
   Color from separation

Reinforced set projection: a new paving algorithm



What we have
$\mathcal{S}_{\mathbb{X}} = \mathcal{C}_{\mathbb{X}}, \mathcal{C}_{\overline{\mathbb{X}}}$ and $\boxed{\mathcal{C}_{\partial \,\mathsf{Proj}\, \mathbb{X}}}$

1. Contraction
2. Color from neighbors
   or
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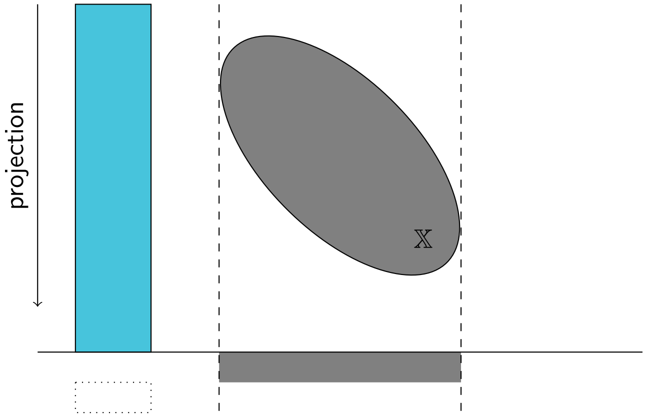

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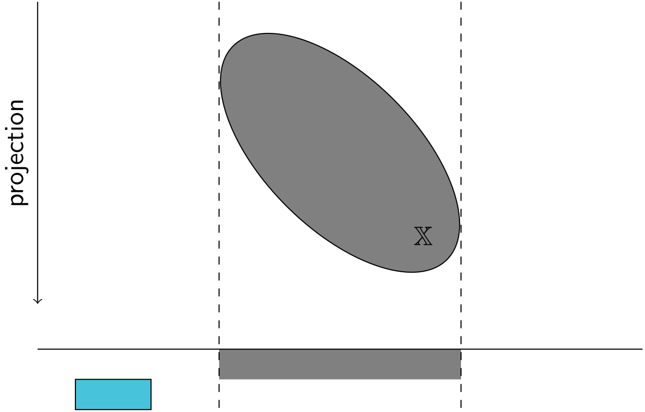

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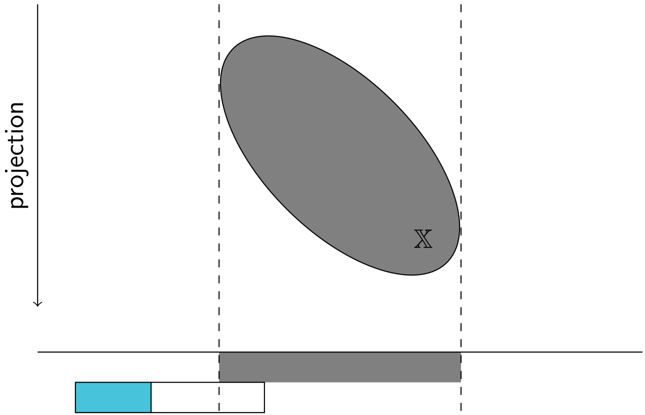

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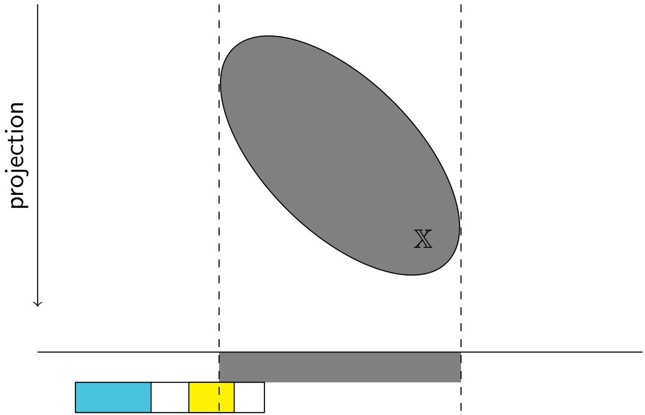

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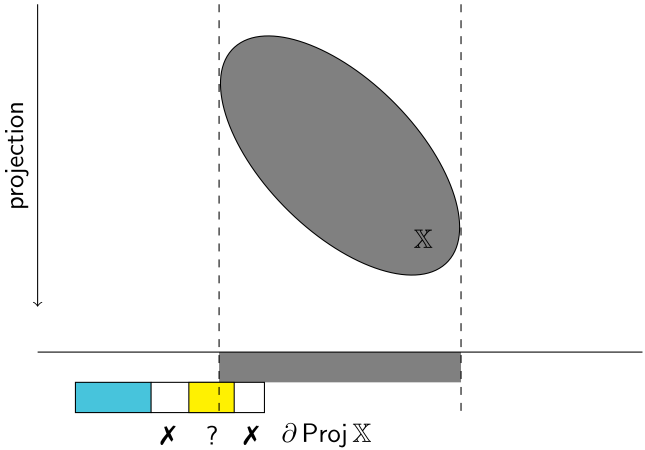

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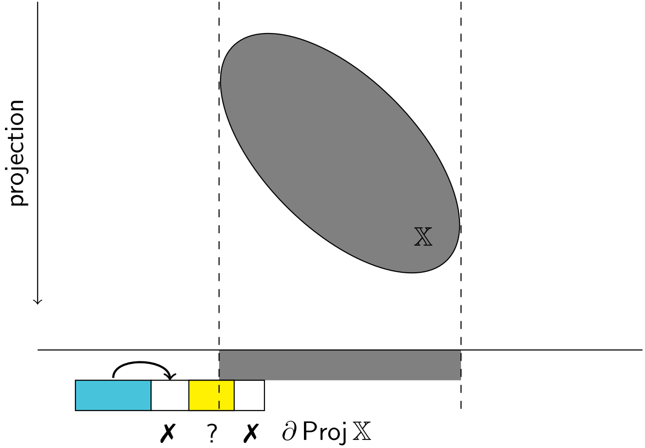

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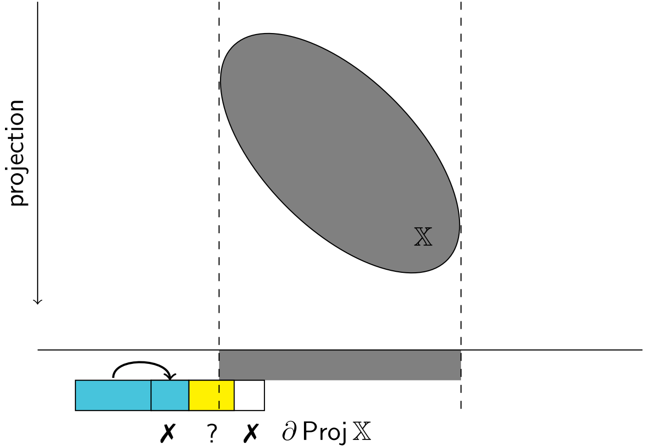

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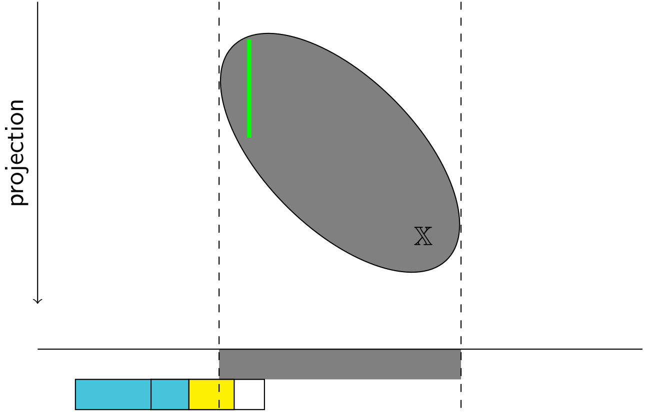

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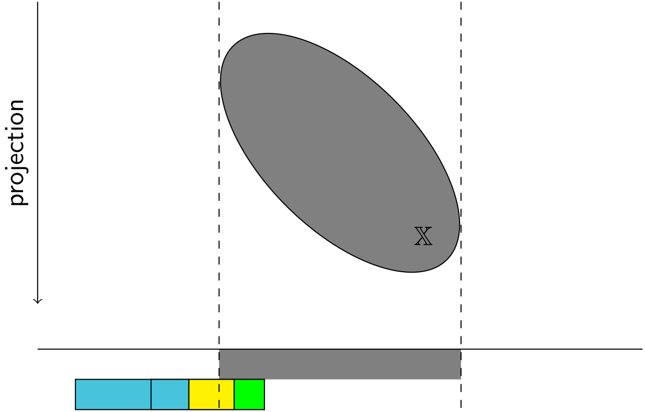

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Usage of projection separators

Using SepProj

$$f(x, y, z) = 2x^2 + 2.2xy + xz + y^2 + z^2 - 10$$

$$\mathbb{X} = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) \leq 0\} \sim \mathcal{S}_{\mathbb{X}}$$

```
from codac import *
```

Using SepProj

$$f(x,y,z) = 2x^2 + 2.2xy + xz + y^2 + z^2 - 10$$

$$\mathbb{X} = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) \leq 0\} \sim \mathcal{S}_{\mathbb{X}}$$

```
from codac import *

# ...

f = Function("x", "y", "z", "2*sqr(x) + 2.2*x*y + x*z + sqr(y) + sqr(z) - 10")
sep_X = SepFunction(f,[-oo,0])
```

Using SepProj

$$f(x,y,z) = 2x^2 + 2.2xy + xz + y^2 + z^2 - 10$$
$$\mathbb{X} = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) \leq 0\} \sim \mathcal{S}_{\mathbb{X}}$$

$$\mathrm{Proj}_{z\in[-10,10]}\,\mathbb{X} = \{(x,y) \in \mathbb{R}^2 \mid z \in [-10,10], (x,y,z) \in \mathbb{X}\} \sim \mathcal{S}_{\mathrm{Proj}_{z\in[-10,10]}\,\mathbb{X}}$$
$$\varepsilon_{xy} = 0.03, \varepsilon_z = 0.015$$

```python
from codac import *

# ...

f = Function("x", "y", "z", "2*sqr(x) + 2.2*x*y + x*z + sqr(y) + sqr(z) - 10")
sep_X = SepFunction(f,[-oo,0])
sep_projX = SepProj(sep_X, Interval(-10, 10), 0.015)
```

Using SepProj

$$f(x,y,z) = 2x^2 + 2.2xy + xz + y^2 + z^2 - 10$$
$$\mathbb{X} = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) \leq 0\} \sim \mathcal{S}_{\mathbb{X}}$$

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$$\varepsilon_{xy} = 0.03, \varepsilon_z = 0.015$$

```python
from codac import *

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f = Function("x", "y", "z", "2*sqr(x) + 2.2*x*y + x*z + sqr(y) + sqr(z) - 10")
sep_X = SepFunction(f,[-oo,0])
sep_projX = SepProj(sep_X, Interval(-10, 10), 0.015)

# ...

SIVIA([[-5,-2],[4.5,7.5]], sep_projX, 0.03)
```
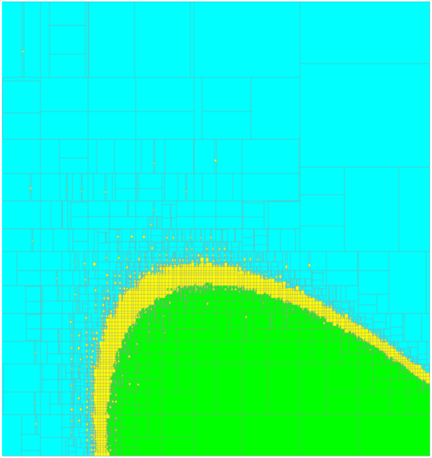

Using SepProj



Execution time: 22 s

Using the new SepProj

$$f(x, y, z) = 2x^2 + 2.2xy + xz + y^2 + z^2 - 10$$

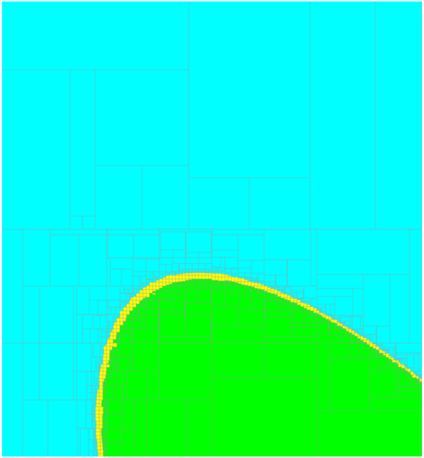
$$\frac{\partial f}{\partial z}(x, y, z) = x + 2z$$

$$\mathcal{S}_{\mathbb{X}}, \mathcal{C}_{\partial \,\mathrm{Proj}\, \mathbb{X}} \longrightarrow \mathcal{S}_{\mathrm{Proj}_{z \in [-10,10]} \mathbb{X}}$$

$$\varepsilon_{xy} = \varepsilon_z = 0.03$$

```
from codac import *

# ...

f = Function("x", "y", "z", "2*sqr(x) + 2.2*x*y + x*z + sqr(y) + sqr(z) - 10")
# ...
sep_projX = NewSepProj(sep_X, ctc_boundary, Interval(-10, 10))

# ...

SIVIA([[-5,-2],[4.5,7.5]], sep_projX, 0.03)
```

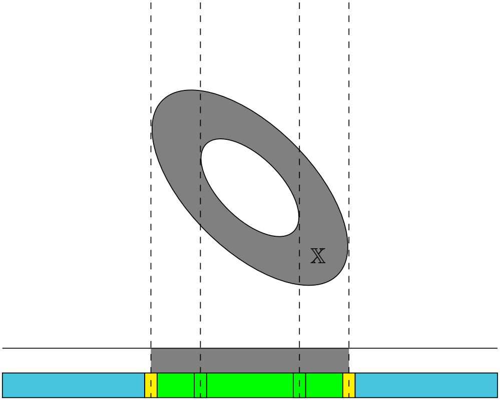
Using the new SepProj

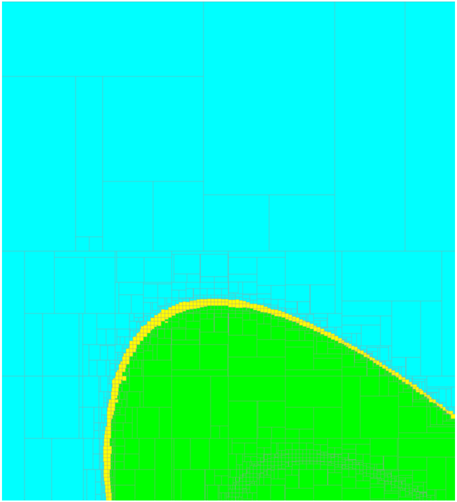


Execution time: 12 s

Fake boundaries

$$\mathbb{X} = \{(x, y, z) \mid -0.3 \leq f(x, y, z) \leq 0\}$$





Execution time: 14 s

Conclusion

Contributions

- $\mathcal{S}_{\mathbb{X}}$ is reinforced with $\mathcal{C}_{\partial\,\mathsf{Proj}\,\mathbb{X}}$
- We proposed a new paving algorithm based on that
- It gets colors from neighboring boxes when it is possible

- It is fast...
- ...but can spend time on fake boundaries

Future work

- Formalize and combine reinforced separators (intersection, union...)