Augmented separators and associated paver

Notations. $[\mathbf{x}]$ is a box. The complement of a set $\mathbb{S}$ is the set of elements not in $\mathbb{S}$ and it is denoted by $\mathbb{S}^{\complement}$.

1 Introduction

1.1 Contractor programming

Let us first recall the formalism of contractor programming [CJ09]. The set of all real intervals is denoted by $\mathbb{IR}$. A Cartesian product of intervals is called a *box*. We use the symbol $[\mathbf{x}]$, surrounded by brackets, for such an object. A *subpaving* of $[\mathbf{x}]$ is a set of non-overlapping boxes included in $[\mathbf{x}]$. A subpaving is either considered as a collection of boxes $\{[\mathbf{x}]^{(1)}, \dots\}$ or as a union $[\mathbf{x}]^{(1)} \cup \dots$ depending on the context. Consequently, a subpaving can either be viewed as a discrete subset of $\mathbb{IR}^n$ or as a compact subset of $\mathbb{R}^n$. A paving of $[\mathbf{x}]$ is a collection of subpavings $\mathbb{K}_1, \dots, \mathbb{K}_N$ such that

$$[\mathbf{x}] = \bigcup_{1 \le k \le N} \bigcup_{[\mathbf{b}] \in \mathbb{K}_k} [\mathbf{b}].$$

For example, Figure 1c shows a paving composed of three subpavings, one with boxes colored in green, one with boxes colored in yellow and one with boxes colored in blue. This representation has been obtained with a paver and some contractors. A *contractor* is an operator from $\mathbb{IR}^n$ into $\mathbb{IR}^n$ which verifies two properties:

$$\begin{array}{l} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \text{ (contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) \text{ (monotonicity).} \end{array} \quad (1)$$

Contractors have been generalized in an algebra [JD14]. Here, we use a set based description of contractors and separators. A set $\mathbb{X}$ is *consistent* with the contractor $\mathcal{C}$ — the relation is denoted by $\mathbb{X} \sim \mathcal{C}$, if

$$\forall [\mathbf{x}],\, \mathcal{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X}. \quad (2)$$

Similarly, a set $\mathbb{X}$ is *consistent* with the separator $\mathcal{S} = \{\mathcal{S}^{in}, \mathcal{S}^{out}\}$ composed of the two contractors $\mathcal{S}^{in}$ and $\mathcal{S}^{out}$ — the relation is denoted by $\mathbb{X} \sim \mathcal{S}$, if

$$\overline{\mathbb{X}} \sim \mathcal{S}^{in} \text{ and } \mathbb{X} \sim \mathcal{S}^{out}. \quad (3)$$

1.2 Projection

We are motivated by the difficulty to get nice projections. In Section 2, we present a paver based on the boundary which is performent in that case.

Some problems are described as the intersection of projections $\pi(\mathbb{S}_1) \cap \pi(\mathbb{S}_2)$, others are described as the projection of an intersection $\pi(\mathbb{S}_1 \cap \mathbb{S}_2)$. We provide tools to compute the boundaries of these sets in Sections 3.1 and 3.2.

2 Paving with boundary contractors

First, we present a new paving algorithm for a set $\mathbb{S}$ based on a contractor consistent with its boundary $\partial\mathbb{S}$ in Section 2. These contractors consistent with the boundary $\partial\mathbb{S}$ of a set $\mathbb{S}$ will be called *boundary contractors*, for conciseness.

We present here a simple yet new set paving algorithm based on a boundary contractor $\mathcal{C}^{\partial}$. Its benefit compared to an algorithm based on a separator $\mathcal{S}$ is visible when it is clear that $\partial\mathcal{S} \subset \mathcal{C}^{\partial}$. In Figure 5 for instance, the paving using $\mathcal{S}_{\pi_z(\mathbb{E})}$ is improved by using $\mathcal{C}_{\partial\pi_z(\mathbb{E})}$, leading to a more precise boundary.

The function used to pave a box $[\mathbf{x}]$ with a certain precision ε is called PAVE. It takes the interval representation of a set in the form of a separator $\mathcal{S}$ or a contractor $\mathcal{C}$, as well as the previously mentioned parameters.

- By calling PAVE($\mathcal{S}$, $[\mathbf{x}]$, ε), where $\mathbb{S} \sim \mathcal{S}$, one gets
    - a list $\mathcal{L}$ of boxes which are contained in $\mathbb{S}$ ($\mathbb{S}^{-} \subset \mathbb{S}$),
    - a list $\mathcal{L}^{\complement}$ of boxes which contain no point of $\mathbb{S}$ ($\mathbb{S}^{\complement^{-}} \subset \mathbb{S}^{\complement}$),
    - and a list $\mathcal{L}^{\Delta}$ of boxes for which we cannot state neither ($\mathbb{S}^{\Delta} \supset \partial\mathbb{S}$).
- By calling PAVE($\mathcal{C}$, $[\mathbf{x}]$, ε), where $\mathbb{S} \sim \mathcal{C}$, one gets
    - a list $\mathcal{L}^{+}$ of boxes which contain $\mathbb{S}$ ($\mathbb{S}^{+} \supset \mathbb{S}$),
    - and a list $\mathcal{L}^{\complement}$ of boxes which contain no point of $\mathbb{S}$ ($\mathbb{S}^{\complement^{-}} \subset \mathbb{S}^{\complement}$),

In Algorithm 1, we present a paving scheme for a set $\mathbb{S}$ which uses a boundary contractor $\mathcal{C}^{\partial} \sim \partial\mathbb{S}$ and an oracle $\mathcal{O}$ telling us whether a set of boxes is inside $\mathbb{S}$, is outside $\mathbb{S}$ or that it is unable to conclude.

- By calling PAVE($\mathcal{C}^{\partial}$, $\mathcal{O}$, $[\mathbf{x}]$, ε), one gets the same behavior as with a separator.

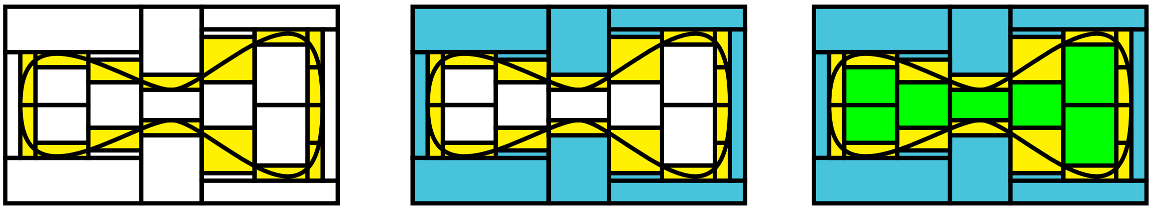
The idea of the algorithm is illustrated by Figure 1. First, the boundary is paved with $\mathcal{C}^{\partial}$, the boxes which contain that set are stored in $\mathcal{L}^{\Delta}$ and the rest is gathered into connected components in $\mathscr{L}$ (see Figure 1a). Each of these components $\mathcal{L}_i$ is either entirely inside or entirely outside the set. The answer is given by the oracle $\mathcal{O}$. If it is inside, the whole set $\mathcal{L}_i$ is added to $\mathcal{L}$ (see Figure 1b). If it is outside, the whole set $\mathcal{L}_i$ is added to $\mathcal{L}^{\complement}$ (see Figure 1c). Otherwise, we were not able to conclude, the set $\mathcal{L}_i$ is thus added to $\mathcal{L}^{\Delta}$.

3 Augmented separators

We then introduce the notion of *augmented separator* in Section 3. It is used to compute boundary contractors for composite sets. Finally, we extend the possibilities by equipping augmented separators with an arithmetic in Section 3.1.

Algorithm 1 Paving of $\mathbb{S}$ using $\mathcal{C}^{\partial}$, such that $\partial\mathbb{S} \sim \mathcal{C}^{\partial}$, and $\mathcal{O}$.

```
Input: A contractor on the boundary $\mathcal{C}^{\partial}$, an oracle $\mathcal{O}$, a box $[\mathbf{x}]$ and a precision value $\varepsilon$
Output: Three lists $\mathcal{L}^{-}$, $\mathcal{L}^{\Delta}$ and $\overline{\mathcal{L}^{-}}$
 1: procedure PAVE($\mathcal{C}^{\partial}$, $\mathcal{O}$, $[\mathbf{x}]$, $\varepsilon$)
 2:     $\mathcal{L}^{\Delta}, \mathcal{L}^{?} \leftarrow$ PAVE($\mathcal{C}^{\partial}, [\mathbf{x}], \varepsilon$)              ▷ Boundary boxes ($\mathcal{L}^{\Delta}$)
 3:     $\mathscr{L} \leftarrow$ separate $\mathcal{L}^{?}$ into connected components     ▷ Connected components ($\mathscr{L}$)
 4:     for $\mathcal{L}_i \in \mathscr{L}$ do
 5:         $\mathbb{U} \leftarrow \bigcup_{[\mathbf{b}] \in \mathcal{L}_i} [\mathbf{b}]$                    ▷ Union of the boxes of $\mathcal{L} \in \mathscr{L}$ ($\mathbb{U}$)
 6:         if the oracle $\mathcal{O}$ tells us that $\mathbb{U}$ is inside $\mathbb{S}$ then
 7:             add $\mathcal{L}_i$ to $\mathcal{L}$
 8:         else if the oracle $\mathcal{O}$ tells us that $\mathbb{U}$ is outside $\mathbb{S}$ then
 9:             add $\mathcal{L}_i$ to $\mathcal{L}^{\complement}$
10:         else
11:             add $\mathcal{L}_i$ to $\mathcal{L}^{\Delta}$
12:         end if
13:     end for
14:     return $\mathcal{L}, \mathcal{L}^{\Delta}, \mathcal{L}^{\complement}$
15: end procedure
```

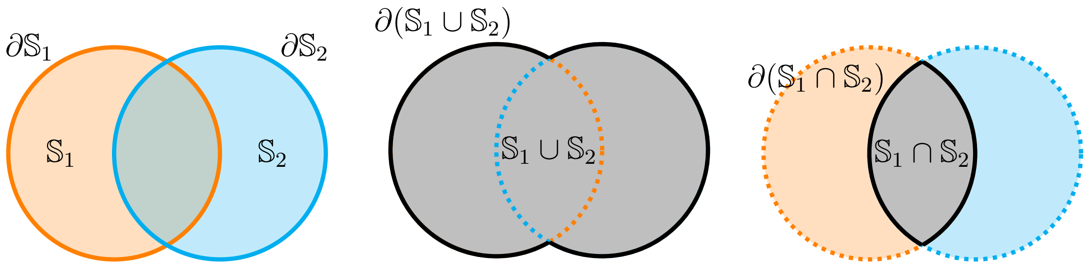


(a) Paving and classification of boundary boxes (Lines 2-3).

(b) Classification of the outside set (Lines 4-12).

(c) Classification of the inside set (Lines 4-12).

Figure 1: Illustration of Algorithm 1.



(a) $\mathbb{S}_1$ and $\mathbb{S}_2$. (b) $\mathbb{S}_1 \cup \mathbb{S}_2$. (c) $\mathbb{S}_1 \cap \mathbb{S}_2$.

Figure 2: Set arithmetic.

3.1 Arithmetic of augmented separators

Definition 3.1. An *augmented separator* $\langle \mathcal{S}, \mathcal{C}^\partial \rangle$ is a couple formed by a separator $\mathcal{S}$ and a contractor $\mathcal{C}^\partial$, such that $\mathbb{S} \sim \mathcal{S}$ and $\partial\mathbb{S} \sim \mathcal{C}^\partial$.

This notion is useful when it is associated with the paving algorithm described in Section 2. We want both $\mathcal{S}$ and $\mathcal{C}^\partial$ to be minimal. However, in projection application we may be able to construct $\mathcal{C}^\partial$ to be very thin/precise but not minimal. We see in Section ? how we can attain minimality in the case of the projection.

We have seen how to get the paving of a set $\mathbb{S}$ represented by an augmented separator $\langle \mathcal{S}, \mathcal{C}^\partial \rangle$ in Section 2. We may also want to get the paving of the union of two sets $\mathbb{S}_1$ and $\mathbb{S}_2$ represented respectively by $\langle \mathcal{S}_1, \mathcal{C}_1^\partial \rangle$ and $\langle \mathcal{S}_2, \mathcal{C}_2^\partial \rangle$. In that case, we need the augmented separator of $\mathbb{S}_1 \cup \mathbb{S}_2$. The construction of such an object is automated thanks to the arithmetic of sets, illustrated by Figure 2. Sets are represented with light colors and their boundaries with solid lines. In Figures 2b and 2c, portions of the original boundaries $\partial\mathbb{S}_1$ and $\partial\mathbb{S}_2$ which do not belong to $\partial(\mathbb{S}_1 \cup \mathbb{S}_2)$, respectively to $\partial(\mathbb{S}_1 \cap \mathbb{S}_2)$, are represented with dotted lines.

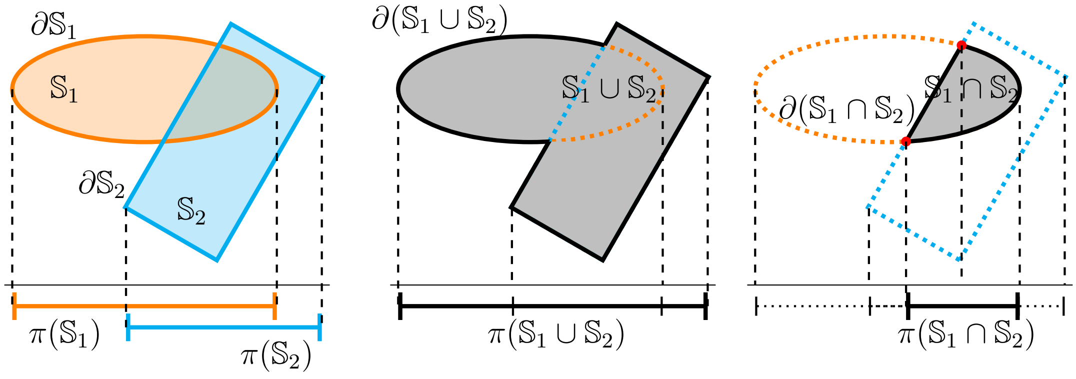
$$\partial(\mathbb{S}_1 \cup \mathbb{S}_2) = \left(\partial\mathbb{S}_1 \cap \mathbb{S}_2^{\complement}\right) \cup \left(\partial\mathbb{S}_2 \cap \mathbb{S}_1^{\complement}\right) \tag{4}$$
$$\partial(\mathbb{S}_1 \cap \mathbb{S}_2) = (\partial\mathbb{S}_1 \cap \mathbb{S}_2) \cup (\partial\mathbb{S}_2 \cap \mathbb{S}_1) \tag{5}$$

For $\mathbb{S} \sim \mathcal{S} = \{\mathcal{S}^{in}, \mathcal{S}^{out}\}$, $\mathbb{S} \sim \mathcal{S}^{out}$ and $\mathbb{S}^{\complement} \sim \mathcal{S}^{in}$. Therefore, the previous topological equalities lead to the following augmented separator arithmetic rules.

$$\langle \mathcal{S}_1, \mathcal{C}_1^\partial \rangle \cup \langle \mathcal{S}_2, \mathcal{C}_2^\partial \rangle = \langle \mathcal{S}_1 \cup \mathcal{S}_2, (\mathcal{C}_1^\partial \cap \mathcal{S}_2^{in}) \cup (\mathcal{C}_2^\partial \cap \mathcal{S}_1^{in}) \rangle \tag{6}$$
$$\langle \mathcal{S}_1, \mathcal{C}_1^\partial \rangle \cap \langle \mathcal{S}_2, \mathcal{C}_2^\partial \rangle = \langle \mathcal{S}_1 \cap \mathcal{S}_2, (\mathcal{C}_1^\partial \cap \mathcal{S}_2^{out}) \cup (\mathcal{C}_2^\partial \cap \mathcal{S}_1^{out}) \rangle \tag{7}$$

3.2 Augmented separators extended for the projection

The set $\mathbb{S}$ that one may want to pave can be the projection of a union, $\pi(\mathbb{S}_1 \cup \mathbb{S}_2)$, or the projection of an intersection, $\pi(\mathbb{S}_1 \cap \mathbb{S}_2)$. Given $\mathbb{S}_1$ and $\mathbb{S}_2$, in order to construct the needed augmented separator one has to call a projection separator for $\mathbb{S}_1 \cup \mathbb{S}_2$, respectively $\mathbb{S}_1 \cap \mathbb{S}_2$, and also construct the contractor on the boundary of that set. We propose to extend augmented separator in order to automate the construction of the latter. For that reason, let us look at the arithmetic of set projections, illustrated by Figure 3. Sets are represented with light colors and their boundaries with solid lines. In Figures 2b and



(a) Projection of $\mathbb{S}_1$ and $\mathbb{S}_2$. (b) $\mathbb{S}_1 \cup \mathbb{S}_2$. (c) $\mathbb{S}_1 \cap \mathbb{S}_2$.

Figure 3: Set projection arithmetic.

3c, portions of the original boundaries $\partial\mathbb{S}_1$ and $\partial\mathbb{S}_2$ which do not belong to $\partial(\mathbb{S}_1 \cup \mathbb{S}_2)$, respectively to $\partial(\mathbb{S}_1 \cap \mathbb{S}_2)$, are represented with dotted lines. The projections of the union and the intersection are presented with solid black segment lines, while the projection of the original sets are represented with thinner dotted segment lines. In order to rewrite $\partial\pi(\mathbb{S}_1 \cup \mathbb{S}_2)$ and $\partial\pi(\mathbb{S}_1 \cap \mathbb{S}_2)$ in Equations 8 and 9, we restrict ourselves to $\mathbb{S}_1$, $\mathbb{S}_2$, $\partial\mathbb{S}_1$, $\partial\mathbb{S}_2$ and $\pi(\cdot)$.

$$\partial\pi(\mathbb{S}_1 \cup \mathbb{S}_2) = \left(\partial\pi(\mathbb{S}_1) \cap \pi\left(\mathbb{S}_2\right)^{\complement}\right) \cup \left(\partial\pi(\mathbb{S}_2) \cap \pi\left(\mathbb{S}_1\right)^{\complement}\right) \tag{8}$$

$$\partial\pi\left(\mathbb{S}_1 \cap \mathbb{S}_2\right) \subset \left(\partial\pi\left(\mathbb{S}_1\right) \cap \mathbb{S}_2\right) \cup \left(\partial\pi\left(\mathbb{S}_2\right) \cap \mathbb{S}_1\right) \cup \pi\left(\partial\mathbb{S}_1 \cap \partial\mathbb{S}_2\right) \tag{9}$$

Remark. We only have an inclusion for Equation 9 and not an equality. It would have required a lifting from the projection.

With the notations from Section 3.1, the previous topological relations (Equations 4 and 5) lead to the following projection augmented separator arithmetic rules (Equations 8 and 9).

$$\begin{aligned}\langle\langle\mathcal{S}_1,\mathcal{C}_1^{\partial}\rangle,\mathcal{C}_1^{\partial\pi}\rangle \cup \langle\langle\mathcal{S}_2,\mathcal{C}_2^{\partial}\rangle,\mathcal{C}_2^{\partial\pi}\rangle = \langle\, &\langle\mathcal{S}_1,\mathcal{C}_1^{\partial}\rangle \cup \langle\mathcal{S}_2,\mathcal{C}_2^{\partial}\rangle\,, \\ &\left(\mathcal{C}_1^{\partial\pi} \cap \mathcal{S}_2^{in}\right) \cup \left(\mathcal{C}_2^{\partial\pi} \cap \mathcal{S}_1^{in}\right))\rangle\end{aligned} \tag{10}$$

$$\begin{aligned}\langle\langle\mathcal{S}_1,\mathcal{C}_1^{\partial}\rangle,\mathcal{C}_1^{\partial\pi}\rangle \cap \langle\langle\mathcal{S}_2,\mathcal{C}_2^{\partial}\rangle,\mathcal{C}_2^{\partial\pi}\rangle = \langle\, &\langle\mathcal{S}_1,\mathcal{C}_1^{\partial}\rangle \cap \langle\mathcal{S}_2,\mathcal{C}_2^{\partial}\rangle\,, \\ &\left(\mathcal{C}_1^{\partial\pi} \cap \mathcal{S}_2^{out}\right) \cup \left(\mathcal{C}_2^{\partial\pi} \cap \mathcal{S}_1^{out}\right) \cup \pi\left(\mathcal{C}_1^{\partial} \cap \mathcal{C}_2^{\partial}\right))\rangle\end{aligned} \tag{11}$$

Due to the remark about Equation 9, in Equation 11 we trade the minimality of the evaluation off against the simplicity of the expression.

3.3 On the disjunctive normal form for contractors

In previous sections, we have given formulas to compute boundary contractors. Note that contractors are not commutative, their order matters. Furthermore, even though they represent sets mathematically, they are actually operators whose result depend on the way there are combined.

In particular, let us look at $\partial\mathbb{S}_1 \cap \partial\mathbb{S}_2$ in Equation 9. Let us assume that $\partial\mathbb{S}_1 = \mathbb{X} \sim \mathcal{C}$, $\partial\mathbb{S}_2 = \mathbb{X}_1 \cup \mathbb{X}_2 \sim \mathcal{C}_1 \cup \mathcal{C}_2$ and that we want to contract a box $[\mathbf{x}]$ with a contractor consistent with $\partial\mathbb{S}_1 \cap \partial\mathbb{S}_2$. Figure 4a shows that situation. Since $\partial\mathbb{S}_1 \cap \partial\mathbb{S}_2 = \mathbb{X} \cap (\mathbb{X}_1 \cup \mathbb{X}_2) = (\mathbb{X} \cap \mathbb{X}_1) \cup (\mathbb{X} \cap \mathbb{X}_2)$, on the one hand we have the conjunctive form

$$\mathbb{X} \cap (\mathbb{X}_1 \cup \mathbb{X}_2) \sim \mathcal{C} \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$$

and on the other hand we have the disjunctive form

$$(\mathbb{X} \cap \mathbb{X}_1) \cup (\mathbb{X} \cap \mathbb{X}_2) \sim (\mathcal{C} \cap \mathcal{C}_1) \cup (\mathcal{C} \cap \mathcal{C}_2)\,.$$

For this example, Figures 4b1 and 4b2 show that computing the union first leads to an overestimation of the contracted box, whereas computing the intersection first contracts directly to a box degenerated to a point, as illustrated by Figures 4c1 and 4c2.

Subsequently, the implementation of Equations 6, 7, 10 and 11 should rewrite intersections in their disjunctive normal form for better contractions.

4 Applications

Reinforcing a separator, with a contractor on the boundary of the set that it represents, is particularly useful in the case of the projection separator. In some cases, the knowledge provided by an efficient contractor on the boundary of the projection can dramatically improve the projection itself.

4.1 Use case of a combined descriptor: projection of a thin ellipsoid

The projection of a thin ellipsoid is an example of a difficult set to pave with the classical paving function. It requires a lot of bisections.

Let us define an ellipsoid $\mathbb{E} = \{(x, y, z) \,|\, f(x, y, z) < 0\}$ with

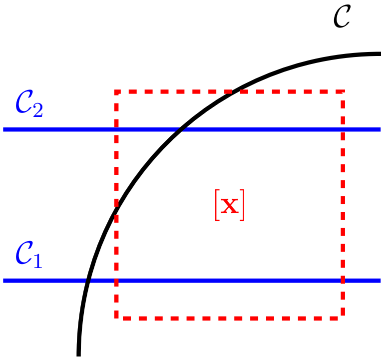
$$f(x, y, z) = 250x^2 + 498xz + y^2 + 250z^2 - 1.$$

We want to observe $\pi_z(\mathbb{E})$ by paving $[-1.2, 1.2]^2$, with a precision of 0.05. So we construct $\mathcal{S}_{\mathbb{E}}$, then we obtain $\mathcal{S}_{\pi_z(\mathbb{E})} = \pi_z(\mathcal{S}_{\mathbb{E}})$ by using a projection separator. The result of the paving is shown in Figure 5b. The computation took 14 s.

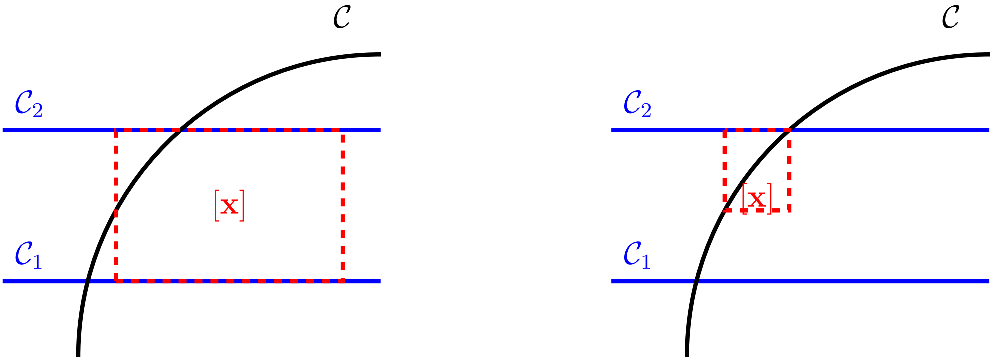
Let us compare that to the paving with a combined descriptor. In addition to $\mathcal{S}_{\pi_z(\mathbb{E})}$, we need a contractor for the boundary of the projection of $\mathbb{E}$. That corresponds to the projection the critical points of $\partial\mathbb{E}$, which verify $f(x, y, z) = 0$ and $\dfrac{\partial f}{\partial z}(x, y, z) = 0$. So, the boundary points of its projection onto the xy-plane verify $g(x, y) = 0$, where $g(x, y) = f(x, y, -\frac{498}{500}x)$. That leads to

$$\begin{cases} \mathbb{E} = \{(x, y, z) \,|\, f(x, y, z) \in [-\infty, 0[\} \\ \partial(\pi_z(\mathbb{E})) = \{(x, y) \,|\, g(x, y) = 0\} \text{ with } g(x, y) = f(x, y, -\frac{498}{500}x). \end{cases}$$

Subsequently, we construct $\mathcal{C}_{\partial\pi_z(\mathbb{E})}$. Thus we have a combined descriptor $\mathcal{C}_{\partial\pi_z(\mathbb{E})} + \mathcal{S}_{\pi_z(\mathbb{E})}$ to pave the space, which gives the result shown in Figure 5b. The computation took 2 s.



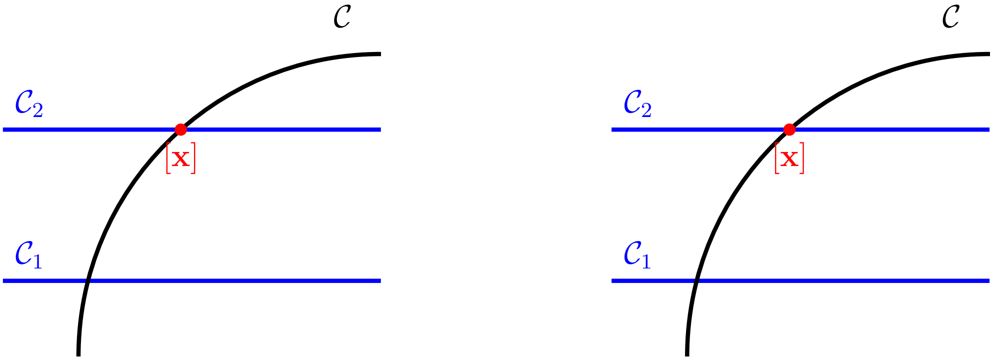
(a) Initial state, before contraction.



(b1) First step: $(\mathcal{C}_1 \cup \mathcal{C}_2)([\mathbf{x}])$.

(b2) Second step: $(\mathcal{C} \cap (\mathcal{C}_1 \cup \mathcal{C}_2))([\mathbf{x}])$.

(b) With a conjunction.

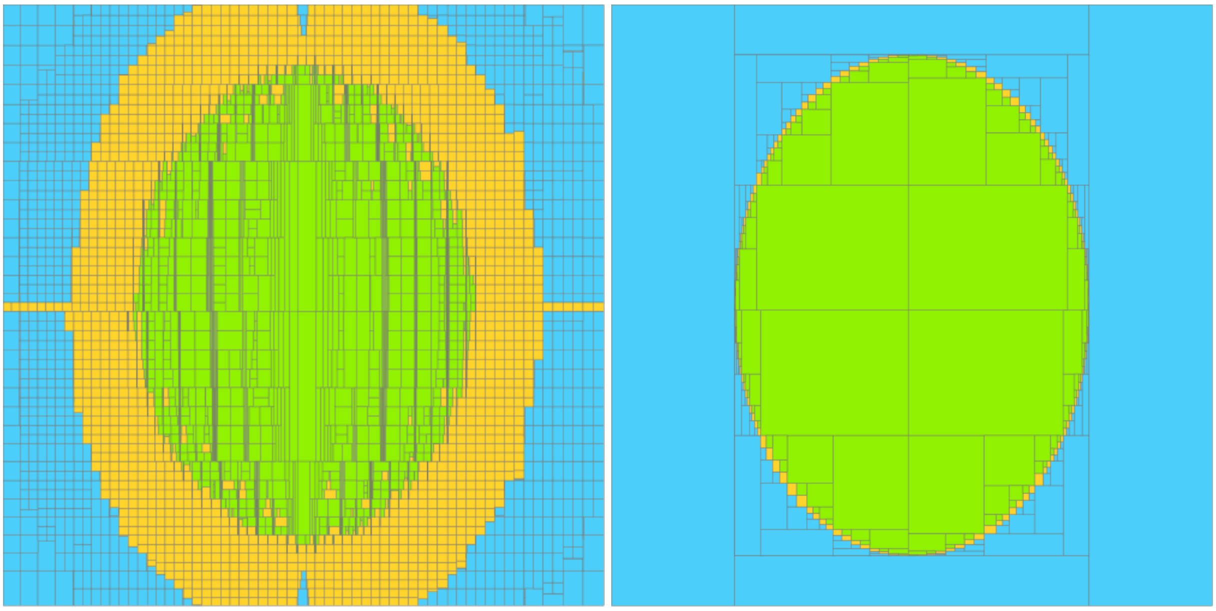


(c1) First step: $(\mathcal{C}_2 \cap \mathcal{C})([\mathbf{x}])$.

(c2) Second step: $((\mathcal{C}_1 \cap \mathcal{C}) \cup (\mathcal{C}_2 \cap \mathcal{C}))([\mathbf{x}])$.

(c) With a disjunction.

Figure 4: Illustration of the benefit of the disjunctive normal form for contractions.



(a) With a separator $\mathcal{S}_{\pi_z(\mathbb{E})}$.

(b) With a boundary contractor $\mathcal{C}^{\partial}_{\pi_z(\mathbb{E})}$.

Figure 5: Projection of a slanted thin ellipsoid.

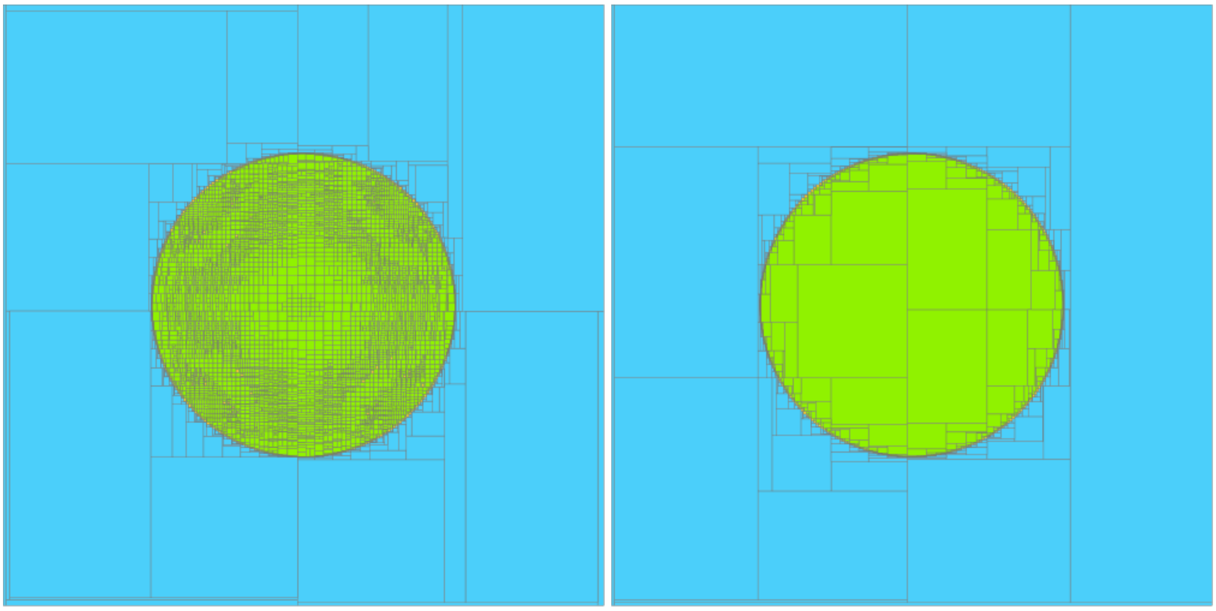
4.2 Use case of a reinforced contractor: hollow ellipsoid

Let us assume that we have a emitter at $(-0.1, 0, 0)$ and a receptor at $(0.1, 0, 0)$. We measure the presence of a vehicle thanks to a signal that traveled (10.0 ± 0.1) m. For a vehicle located at (x, y, z), the signal travels a distance given by $f(x, y, z) = \sqrt{(x+0.1)^2 + y^2 + z^2} + \sqrt{(x-0.1)^2 + y^2 + z^2}$. The possible positions of the vehicle in $\mathbb{R}^3$ belong to the set

$$\mathbb{S} = \{(x, y, z) \mid f(x, y, z) \in [9.9, 10.1]\}.$$

That is a very thin holllow ellipsoid, close to a sphere. We are only interested in the x- and y-coordinates, that is to say that we want the projection parallel to the z-axis. We have constructed $\mathcal{S}_{\pi_z(\mathbb{S})}$ and paved $[-10, 10]^2$ with a precision of 0.1. The result is shown in Figure 6a. The computation took 36 s.

Once again, let us compare that to the paving with a combined descriptor. In addition to $\mathcal{S}_{\pi_z(\mathbb{S})}$, we need a contractor for the boundary of the projection of $\mathbb{S}$. That generally corresponds to the projection the critical points of $\partial\mathbb{S}$, which verify $f(x, y, z) = 9.9$ or $f(x, y, z) = 10.1$ and $\dfrac{\partial f}{\partial z}(x, y, z) = 0$. We construct $\mathcal{C}_{\partial\pi_z(\mathbb{S})}$ accordingly. Here, the points verifying $f(x, y, z) = 9.9$ and $\dfrac{\partial f}{\partial z}(x, y, z) = 0$ actually contribute to what we called a fake boundary. Subsequently, we construct a reinforced boundary contractor $\mathcal{C}^{+}_{\partial\pi_z(\mathbb{S})} = \{\mathcal{C}_{\partial\pi_z(\mathbb{S})}, \mathcal{S}_{\pi_z(\mathbb{S})}\}$. With, the combined descriptor $\mathcal{C}^{+}_{\partial\pi_z(\mathbb{S})} + \mathcal{S}_{\pi_z(\mathbb{S})}$, we get the result shown in Figure 6b. The computation took 17 s.



(a) With a separator $\mathcal{S}_{\pi_z(\mathbb{S})}$.

(b) With a boundary contractor $\mathcal{C}^{+}_{\partial\pi_z(\mathbb{S})}$.

Figure 6: Location of a vehicle.

References

[CJ09] Gilles Chabert and Luc Jaulin. Contractor programming. *Artificial Intelligence*, 173(11):1079–1100, 2009.

[JD14] Luc Jaulin and Benoît Desrochers. Introduction to the algebra of separators with application to path planning. *Engineering Applications of Artificial Intelligence*, 33:141–147, 2014.